

HETEROCLINIC CYCLES ARISING IN GENERIC UNFOLDINGS OF NILPOTENT SINGULARITIES

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ABSTRACT. In this paper we study the existence of heteroclinic cycles in generic unfoldings of nilpotent singularities. Namely we prove that any nilpotent singularity of codimension four in \mathbb{R}^4 unfolds generically a bifurcation hypersurface of bifocal homoclinic orbits, that is, homoclinic orbits to equilibrium points with two pairs of complex eigenvalues. We also prove that any nilpotent singularity of codimension three in \mathbb{R}^3 unfolds generically a bifurcation curve of heteroclinic cycles between two saddle-focus equilibrium points with different stability indexes. Under generic assumptions these cycles imply the existence of homoclinic bifurcations. Homoclinic orbits to equilibrium points with complex eigenvalues are the simplest configurations which can explain the existence of complex dynamics as, for instance, strange attractors. The proof of the arising of these dynamics from a singularity is a very useful tool, particularly for applications.

1. INTRODUCTION

The relationship between dynamic complexity and the presence of homoclinic orbits was discovered by Poincaré more than a century ago. In his famous essay on the stability of the solar system [45], Poincaré showed that the invariant manifolds of a hyperbolic fixed point of a diffeomorphism could cut each other at points, called homoclinics, which yield the existence of more and more points of this type and consequently, a very complicated configuration of the manifolds. Many years later, Birkhoff [6] showed that, in general, near a homoclinic point there exists an extremely intricated set of periodic orbits, mostly with a very long period. By the mid 60's, Smale [54] placed his geometrical device, the Smale horseshoe, in a neighborhood of a transversal homoclinic point. The horseshoes explained the Birkhoff's result and arranged the complicated dynamics that occur near a homoclinic orbit by means of a conjugation to the Bernoulli's shift. In [40] authors proved the appearance of strange attractors during the process of creation or destruction of the Smale horseshoes which appear through a bifurcation of a tangential homoclinic point. These attractors are like those shown in [2] for the Hénon family, that is, they are nonhyperbolic and persistent in the sense of measure.

In the framework of vector fields, Shil'nikov [52] proved that in every neighborhood of a homoclinic orbit to a hyperbolic equilibrium point of an analytical vector field on \mathbb{R}^3 , with eigenvalues λ and $-\varrho \pm \omega i$ such that $0 < \varrho < \lambda$, that is, the so-called *Shil'nikov homoclinic orbit*, there exists a countable set of periodic orbits. This result is similar to that found by Birkhoff for diffeomorphisms and thus, it should be understood in a manner similar to that devised by Smale. Indeed, Tresser [55] showed that in every neighborhood of such a homoclinic

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orbit, an infinity of linked horseshoes can be defined in such a way that the dynamics is conjugated to a subshift of finite type on an infinite number of symbols. Once again, these horseshoes appear and disappear by means of generic homoclinic bifurcations leading to persistent non hyperbolic strange attractors like those in [40].

As follows from [41], nonhyperbolic dynamics is dense in the space \mathcal{X} of vector fields with a Shil'nikov homoclinic orbit. In particular, for each $\varepsilon > 0$, the subset of vector fields with a homoclinic tangency to a hyperbolic periodic orbit in an ε -neighbourhood of the homoclinic orbit is dense in \mathcal{X} . These tangencies give rise to suspended Hénon-like strange attractors. In [46, 47] it was proved that infinitely many of these strange attractors can coexist in non generic families of vector fields with a Shil'nikov homoclinic orbit, for parameter values in a set of positive Lebesgue measure. Later [25], it was proved that an infinity of such attractors can coexist in a more general context. For an extensive study of the phenomena accompanying homoclinic bifurcations, see [4, 26, 48].

Because of the importance of homoclinic orbits in Dynamics, many papers were devoted to prove their existence. A seminal work was due to Melnikov [37], who introduced original ideas to prove the existence of transversal homoclinic orbits in non-autonomous perturbations of a planar hamiltonian vector field. These ideas were developed in [10] in order to determine both, homoclinic bifurcation curves and the existence of subharmonics in two-parameter families of non-autonomous second order differential equations. In [43], Palmer developed a theory involving transversal homoclinic points and exponential dichotomies that was very useful for the study of homoclinic bifurcations in higher dimensions.

Since Shil'nikov homoclinic orbits are not transversal, Melnikov's techniques had to be modified in order to prove their existence in families of vector fields. In [49], generic families of quadratic three dimensional vector fields with Shil'nikov homoclinic orbits were given. Putting together ideas from [49, 10, 43], it was proved in [28] that Shil'nikov homoclinic orbits appear in generic unfoldings of a nilpotent singularity of codimension four in \mathbb{R}^3 . Since singularities are the simplest elements to be found in phase portraits of vector fields, arguing the existence of homoclinic orbits from the presence of singularities is a highly relevant task. Nevertheless, in order to get the greatest interest in applications, such singularities should be of codimension as low as possible. With this in mind, the result obtained in [28] was improved in [29], where it was showed that

Theorem A. *Shil'nikov homoclinic orbits appear in every generic unfolding of the nilpotent singularity of codimension three in \mathbb{R}^3 .*

Proving that Shil'nikov homoclinic orbits can be unfolded generically from a singularity of codimension less than three is currently a very interesting open problem. The dimension of the center manifold should be at least three. The lowest codimension singularities in \mathbb{R}^3 with a 3-dimensional center manifold are the Hopf-zero singularities which have codimension two [22]. The difficulties that appear on studying the existence of Shil'nikov homoclinic orbits in generic unfoldings of Hopf-zero singularities are discussed in [17].

Theorem A was essential in [18] to prove the existence of persistent strange attractors in the four parametric family of vector fields obtained when two Brusselators are linearly coupled by diffusion. Indeed, this family is a generic unfolding of three-dimensional nilpotent singularities of codimension three. Therefore it displays Shil'nikov homoclinic orbits and, consequently, persistent strange attractors. Nevertheless, this family may display a richer dynamics. Three-dimensional nilpotent singularities appear along two bifurcation curves which emerge from a bifurcation point corresponding to a four-dimensional nilpotent singularity of codimension four, for which the family is also a generic unfolding. Therefore, one should wonder whether a different class of homoclinic orbits can take place from this four-dimensional nilpotent singularity. In this paper we will prove the following result:

Theorem B. *In every generic unfolding of a four-dimensional nilpotent singularity of codimension four there is a bifurcation hypersurface of homoclinic orbits to equilibrium points with two pairs of eigenvalues $\rho_k \pm \omega_k i$, with $k = 1, 2$, such that $\rho_1 < 0 < \rho_2$.*

Homoclinic orbits in Theorem B are usually known as *bifocal homoclinic orbits* or, shortly, *bifocus*. Shil'nikov [53] was again the first one in studying the dynamics associated to them. He proved, as in [52], the existence of a countable set of periodic orbits in the non-resonant case $-\rho_1 \neq \rho_2$. Subsequent works [13, 21, 33, 23] were devoted to analyze the formation and bifurcations of these periodic orbits by studying the Poincaré map associated to the flow in a neighborhood of the bifocus. Devaney [13] considers the hamiltonian case, hence with $-\rho_1 = \rho_2$. He proves that for any local transverse section to the homoclinic orbit, and for any positive integer N , there is a compact invariant hyperbolic set on which the Poincaré map is conjugate to the Bernoulli shift on N symbols. In seeking to determine the invariant set of this Poincaré map in the general case, it is shown in [21] that this set is contained in a neighborhood of a spiral sheet (shaped like a scroll). In fact, the invariant set is a neighborhood of the intersection of this scroll and its image under the map, which is another scroll, in general skewed and offset from the original. In [33] the authors extend the known theory regarding bifocal homoclinic bifurcations and present numerical verification of the more interesting theoretical predictions that had been made. Härterich [23] studies bifocal homoclinic orbits arising in reversible systems, hence again with $-\rho_1 = \rho_2$. He proves that for any $N \geq 2$ there exists infinitely many N -homoclinic orbits in a neighborhood of the primary homoclinic orbit. Each of them is accumulated by one or more families of N -periodic orbits.

As for Shil'nikov homoclinic orbits, it has been proved (see [42]) that homoclinic tangencies to hyperbolic periodic orbits are dense in the space of vector fields with a bifocal homoclinic orbit. Nevertheless, despite the abundant literature regarding bifocus, as far as we know, no result has been established relating the existence of bifocal homoclinic bifurcations with the existence of persistent strange attractors. This, in spite of a bifocus seems to be a scenario for more complicated dynamics than those inherent to Shil'nikov homoclinic orbits, where the existence of such strange attractors has been proved. In fact it seems natural to think that the dynamical complexity associated with homoclinic cycles increases with dimension. For instance, strange attractors with more than one positive Lyapunov exponent could appear.

Therefore, bearing in mind a possible extension of Theorem B to higher dimensions, we will begin its proof working with n -dimensional nilpotent singularities of codimension n .

Let X be a C^∞ vector field on \mathbb{R}^n with $X(0) = 0$ and 1-jet linearly conjugated to $\sum_{k=1}^{n-1} x_{k+1} \partial / \partial x_k$. Introducing appropriate coordinates, X can be written as

$$\sum_{k=1}^{n-1} x_{k+1} \frac{\partial}{\partial x_k} + f(x_1, \dots, x_n) \frac{\partial}{\partial x_n},$$

with $f(x) = O(\|x\|^2)$ where $x = (x_1, \dots, x_n)$. It is said that X is a *nilpotent singularity of codimension n* if the generic condition $\partial^2 f / \partial x_1^2 \neq 0$ is fulfilled. As we will explain in Section 3, in appropriate coordinates and after rescaling, any generic n -parametric unfolding X_λ of a nilpotent singularity can be written in a neighborhood of the origin as

$$\sum_{k=1}^{n-1} y_{k+1} \frac{\partial}{\partial y_k} + (v_1 + \sum_{k=2}^n v_k y_k + y_1^2 + O(\varepsilon)) \frac{\partial}{\partial y_n}, \quad (1.1)$$

with $\varepsilon > 0$ and $v_1^2 + \dots + v_n^2 = 1$. The limit family obtained for $\varepsilon = 0$, will play a main role. It is time reversible with respect to the involution $R(y_1, y_2, \dots, y_n) = ((-1)^n y_1, (-1)^{n-1} y_2, \dots, -y_n)$ for parameter values on the set

$$\mathcal{T} = \{(v_1, \dots, v_n) \in \mathbb{S}^{n-1} : v_{n-2i} = 0 \text{ with } i = 0, \dots, \lfloor (n-2)/2 \rfloor\},$$

where $\lfloor \cdot \rfloor$ denotes the floor function. This manifold \mathcal{T} of dimension $\lfloor n/2 - 1 \rfloor$ is called the reversibility set of the n -dimensional nilpotent limit family.

For $n = 4$ and for values of the parameters $v_1 < 0$, $v_2 = v_4 = 0$, the limit family can be transformed in

$$x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} + (-x_1 + \eta_3 x_3 + x_1^2) \frac{\partial}{\partial x_4}, \quad (1.2)$$

where $\eta_3 = 2^{-1/2}(-v_1)^{-1/4}v_3$. Denoting $u = x_1$ and $P = -\eta_3$ the vector field (1.2) is equivalent to the fourth-order equation $u^{(iv)}(t) + Pu''(t) + u(t) - u^2(t) = 0$, which has been widely studied [1, 12, 3, 8] due to its role in some applications as the study of travelling waves of the Korteweg- de Vries equation

$$u_t = u_{xxxx} - bu_{xxx} + 2uu_x,$$

or the description of the displacement of a compressed strut with bending softness resting on a nonlinear elastic foundation [9]. In particular, according to [1], when $\eta_3 = 2$ the vector field (1.2) has a homoclinic orbit to a hyperbolic equilibrium point at which the linear part has a pair of double real eigenvalues ± 1 . We will complete the proof of Theorem B by proving that the homoclinic orbit persists for parameter values on a hypersurface $\mathcal{H}om$ which intersects $\varepsilon > 0$ and is obtained by studying the appropriate bifurcation equation. Moreover we will show that $\mathcal{H}om$ contains regions corresponding to bifocal homoclinic orbits. An essential fact used in [1] to prove the existence of homoclinic orbits in (1.2) is that it is a family of hamiltonian vector fields. This permits to apply the general theory developed in [24]. Again bearing in mind a possible extension of Theorem B to higher dimensions we will prove that, for any even n and for parameter values on the reversibility set, the vector fields in the limit family of (1.1) are hamiltonian.

Methods used in the proof of Theorem B also allow us to prove the existence of topological Bykov cycles, which will be defined below, in the case $n = 3$, according with the following result.

Theorem C. *In every generic unfolding of a three-dimensional nilpotent singularity of codimension three there is a bifurcation curve of topological Bykov cycles.*

For $n = 3$ and when $v_1 < 0$ and $v_2 < 0$, the family (1.1) can be transformed into the family

$$x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + (c^2 - x_2 + vx_3 - x_1^2/2 + O(\varepsilon)) \frac{\partial}{\partial x_3} \quad (1.3)$$

where $c^2 = 2v_1/v_2^3$ and $v = v_3/(-1/v_2)^{1/2}$. When $v = 0$ this limit family is equivalent to the third-order equation $x'''(t) + x'(t) + x(t)^2/2 = c^2$, which has been studied extensively in the literature (see [31, 32, 38] and references therein) because it plays a very relevant role in the study of the existence of steady solutions and travelling waves of the Kuramoto-Shivashinsky $u_t + u_{xxx} + u_{xx} + u_x^2/2 = 0$. For each value of c , family (1.3) has two saddle-focus equilibria P_{\pm} . In particular, the eigenvalues λ and $-\rho \pm i\omega$ at P_{-} satisfy that $0 < \rho < \lambda$, that is, the spectral assumptions in Shil'nikov's theorem. In [31] it is proved the existence of a heteroclinic connection $\Gamma_1 = W^u(P_{-}) \cap W^s(P_{+})$ when $c = c_k = 15\sqrt{22/19^3}$. For this same value of c , it was proved in [29] the existence of a topologically transverse intersection Γ_2 between the two-dimensional invariant manifolds $W^u(P_{+})$ and $W^s(P_{-})$. Therefore, $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \{P_{-}, P_{+}\}$ is a heteroclinic cycle, known in the literature as *Bykov cycle*. Bifurcations arising from the breaking of this codimension two heteroclinic cycle have been widely studied in the literature [15, 20]. In particular, the birth of homoclinic orbits was considered in [15] for the case which concerns us. It follows from these papers that in any generic 2-parametric unfolding of a Bykov cycle there exist two homoclinic bifurcation curves, which spiral in to the Bykov cycle bifurcation point. Genericity can be read in terms of the transversality between the 2-dimensional invariant manifolds, which can be replaced by the condition of topological transversality, and the generic splitting of the connection along the 1-dimensional invariant manifolds (details can be seen in [15, 20, 29]).

In this paper we will prove that family (1.3) exhibits topological Bykov cycles for parameters along a curve $\gamma = \{(c(\varepsilon), v(\varepsilon), \varepsilon) : \varepsilon \in [0, \varepsilon_0)\}$ for some $\varepsilon_0 > 0$ with $c(0) = c_k$ and $v(0) = 0$. Fixing a section Σ transverse to Γ_1 we consider the splitting function $h(c, v, \varepsilon) = (h_1(c, v, \varepsilon), h_2(c, v, \varepsilon))$ defined as the distance between $W^u(P_{-}(\tau)) \cap \Sigma$ and $W^s(P_{+}(\tau)) \cap \Sigma$ for $\tau = (c, v, \varepsilon)$ close enough to $\tau_k = (c_k, 0, 0)$. Note that h take values on \mathbb{R}^2 and existence of Bykov cycle is equivalent to $h(c, v, \varepsilon) = (0, 0)$. We will prove that

$$\begin{vmatrix} \frac{\partial h_1}{\partial c}(\tau_k) & \frac{\partial h_1}{\partial v}(\tau_k) \\ \frac{\partial h_2}{\partial c}(\tau_k) & \frac{\partial h_2}{\partial v}(\tau_k) \end{vmatrix} \neq 0. \quad (1.4)$$

Hence the existence of γ , and therefore Theorem C, follows from the Implicit Function Theorem.

We should remark that the above generic condition guarantees the existence of Shil'nikov homoclinic orbits, in the sense of Theorem A. Indeed, for each $\varepsilon > 0$ small enough and fixed, (1.4) implies that the splitting function is a local diffeomorphism or, in other words, that the splitting of the connection along the 1-dimensional invariant manifolds is generic. Therefore there exists a Shil'nikov bifurcation surface shaped as a scroll around γ . In the proof of Theorem A given in [29], for the sake of brevity, we did not include the computation of the above generic condition, although we appealed to it. Since the publication of that paper such computation has been frequently demanded to us.

The paper is organized as follows. In §2 we include a brief summary of results about dichotomies in order to get a precise formulation of the bifurcation equations which are required in the subsequent sections to prove the existence of heteroclinic and homoclinic orbits. §3 is devoted to introduce nilpotent singularities on \mathbb{R}^n and the limit families obtained after rescaling properly a generic unfolding. There we also state that when n is an even number and the parameter values belong to the reversibility set, the limit family consists of hamiltonian vector fields. Theorem B and Theorem C are proved in §4 and §5, respectively.

2. DICHOTOMIES AND BIFURCATION EQUATIONS.

Let $x' = f(x)$ be a nonlinear equation, where $x \in \mathbb{R}^n$ and f is a regular enough vector field, and assume that it has a heteroclinic orbit $\gamma = \{p(t) : t \in \mathbb{R}\}$ connecting two hyperbolic equilibrium points p_+ and p_- (if $p_+ = p_-$, γ is said homoclinic). Consider a family

$$x' = f(x) + g(\lambda, x), \quad (2.1)$$

with $\lambda \in \mathbb{R}^k$ and g regular enough, such that $g(0, x) = 0$. For any λ small enough, family (2.1) has hyperbolic equilibrium points $p_+(\lambda)$ and $p_-(\lambda)$, continuation of p_+ and p_- , respectively, and the stability index is preserved. In order to study the persistence of the heteroclinic orbit for λ small enough we introduce the change of variables $x(t) = z(t) + p(t)$ in (2.1) to obtain

$$z'(t) = Df(p(t))z(t) + b(\lambda, t, z(t)), \quad (2.2)$$

where

$$b(\lambda, t, z(t)) = f(p(t) + z(t)) - f(p(t)) - Df(p(t))z(t) + g(\lambda, p(t) + z(t)).$$

Notice that $b(0, t, 0) = D_z b(0, t, 0) = 0$ for all $t \in \mathbb{R}$.

Persistence of heteroclinic orbits in (2.1) implies the existence of bounded solutions for (2.2) which, in turn, implies the existence of bounded solutions for a equation as

$$z'(t) = Df(p(t))z(t) + b(t), \quad (2.3)$$

where b belongs to the space $C_b^0(\mathbb{R}, \mathbb{R}^n)$. In the sequel $C_b^k(\mathbb{R}, \mathbb{R}^n)$ denotes the Banach space of bounded continuous \mathbb{R}^n -valued functions whose derivatives up to order k exist and are bounded and continuous. The existence of bounded solutions of a linear equation $x' = A(t)x + b(t)$, as that in (2.3), will be given in terms of exponential dichotomies of the homogeneous equation $x' = A(t)x$ and its adjoint $w' = -A(t)^*w$, where $A(t)^*$ denotes the conjugate transpose of $A(t)$. The classical references for the study of exponential dichotomies are [39, 11, 43, 44].

2.1. Exponential dichotomy. Let $X(t)$ be a fundamental matrix of

$$x' = A(t)x, \quad x \in \mathbb{R}^n, \quad (2.4)$$

where $A(t)$ is defined and continuous on an interval $J \subseteq \mathbb{R}$.

Definition 2.1. It is said that the equation (2.4) has an exponential dichotomy on J if there exists a projection $P : \mathbb{R}^n \rightarrow \mathbb{R}^n$, that is, an n by n matrix P with $P^2 = P$, and positive constants K, L, α and β such that for every $s, t \in J$,

$$\begin{aligned} \|X(t)PX^{-1}(s)\| &\leq Ke^{-\alpha(t-s)} \quad \text{for } t \geq s, \\ \|X(t)(I - P)X^{-1}(s)\| &\leq Le^{-\beta(s-t)} \quad \text{for } s \geq t. \end{aligned} \quad (2.5)$$

Let us define $\mathcal{P}(s) = X(s)PX^{-1}(s)$ for each $s \in J$. Notice that, according with the above definition, $\mathcal{P}(s)$ is the projection corresponding to the fundamental matrix $Y(t) = X(t)X^{-1}(s)$ of (2.4) and we can give an alternative definition of exponential dichotomy.

Definition 2.2. It is said that the equation (2.4) has an exponential dichotomy on J if for all $s \in J$ there exists a projection $\mathcal{P}(s) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and positive constants K, L, α and β independent of s such that for all $t \in J$ the matrix $X^{-1}(t)\mathcal{P}(t)X(t)$ has constant coefficients and

$$\begin{aligned} \|X(t)X^{-1}(s)\mathcal{P}(s)\| &\leq Ke^{-\alpha(t-s)} \quad \text{for all } t \geq s, \\ \|X(t)X^{-1}(s)(I - \mathcal{P}(s))\| &\leq Le^{-\beta(s-t)} \quad \text{for all } s \geq t. \end{aligned}$$

Although the notion of exponential dichotomy is stated for any $J \subseteq \mathbb{R}$, the most interesting cases are when J is not bounded. We are particularly interested in $J = [\tau, \infty)$ or $J = (-\infty, \tau]$. In such cases the notions of stable and unstable subspaces can be introduced in terms of the ranges of the projections of the exponential dichotomies.

Definition 2.3. Suppose that the matrix $A(t)$ in (2.4) is defined and continuous on $J = [\tau, \infty)$ (resp. $J = (-\infty, \tau]$). For each $t_0 \in J$ the stable (resp. unstable) subspace for initial time $t = t_0$ is defined as

$$\begin{aligned} E_{t_0}^s &= \{\xi \in \mathbb{R}^n : \|X(t)X^{-1}(t_0)\xi\| \rightarrow 0 \text{ when } t \rightarrow \infty\} \\ (\text{resp. } E_{t_0}^u &= \{\xi \in \mathbb{R}^n : \|X(t)X^{-1}(t_0)\xi\| \rightarrow 0 \text{ when } t \rightarrow -\infty\}). \end{aligned}$$

Below we give a collection of results which can be helpful to follow the paper. Their proofs are available in the literature.

Proposition 2.4. Suppose that the equation $x' = A(t)x$ has an exponential dichotomy on J .

i) When $J = [\tau, \infty)$, $E_{t_0}^s$ coincides with the range $\mathcal{R}(\mathcal{P}(t_0))$ of $\mathcal{P}(t_0)$ for all $t_0 \in J$. Furthermore

$$\mathcal{R}(\mathcal{P}(t_0)) = \{\xi \in \mathbb{R}^n : \sup_{t \geq t_0} \|X(t)X^{-1}(t_0)\xi\| < \infty\},$$

and for all $t_0, t_1 \in J$ it follows that $E_{t_1}^s = X(t_1)X^{-1}(t_0)E_{t_0}^s$.

ii) When $J = (-\infty, \tau]$, $E_{t_0}^u$ coincides with the kernel $\mathcal{N}(\mathcal{P}(t_0))$ of $\mathcal{P}(t_0)$ for all $t_0 \in J$. Furthermore

$$\mathcal{N}(\mathcal{P}(t_0)) = \{\xi \in \mathbb{R}^n : \sup_{t \leq t_0} \|X(t)X^{-1}(t_0)\xi\| < \infty\},$$

and for all $t_0, t_1 \in J$ it follows that $E_{t_1}^u = X(t_1)X^{-1}(t_0)E_{t_0}^u$.

From the above proposition it follows that the linear flow sends $E_{t_0}^s$ and $E_{t_0}^u$ to $E_{t_1}^s$ and $E_{t_1}^u$, respectively. Accordingly, once $E_{t_0}^s$ and $E_{t_0}^u$ are fixed, the stable and unstable subspaces are determined for all t . Therefore, the projections are also determined for each $t \in J$ once they are defined for $t = t_0$. The same observation follows taking into account the uniqueness of solutions for the equation

$$\mathcal{P}'(s) = X'(s)PX^{-1}(s) + X(s)P(X^{-1}(s))' = A(s)\mathcal{P}(s) - \mathcal{P}(s)A(s).$$

Lemma 2.5. *If the linear homogeneous equation $x' = A(t)x$, with $t \in (-\infty, \infty)$, has exponential dichotomy $[\tau, \infty)$ (resp. $(-\infty, \tau]$) for some $\tau \in \mathbb{R}$ then it has exponential dichotomy on $[t_0, \infty)$ (resp. $(-\infty, t_0]$) for all $t_0 \in \mathbb{R}$.*

The next result [44, Lemma 7.4] states that exponential dichotomy is a robust property with respect to small enough perturbations of $A(t)$.

Proposition 2.6. *Suppose that $x' = A(t)x$ has an exponential dichotomy on $J = [a, b]$ (with $-\infty \leq a < b \leq \infty$) with projection matrix function $\mathcal{P}(t)$, with constants K_1, K_2 and exponents α_1, α_2 . Let β_1 and β_2 be such that $0 < \beta_1 < \alpha_1$ and $0 < \beta_2 < \alpha_2$. Then there exists $\delta_0 = \delta_0(K_1, K_2, \alpha_1, \alpha_2, \beta_1, \beta_2) > 0$ such that if $B(t)$ is a continuous matrix function with*

$$\|B(t)\| \leq \delta_t \leq \delta_0 \quad \text{for all } t \in J,$$

the perturbed system

$$x' = [A(t) + B(t)]x$$

has an exponential dichotomy on J with constants L_1, L_2 exponents β_1, β_2 and projection matrix $\mathcal{Q}(t)$ satisfying that

$$\|\mathcal{Q}(t) - \mathcal{P}(t)\| \leq N\delta_t,$$

where L_1, L_2 and N are constants which only depend on K_1, K_2, α_1 and α_2 .

From the above result and Lemma 2.5 it follows the existence of an exponential dichotomy for the homogeneous part $z' = Df(p(t))z$ of the equation (2.3). Since

$$\lim_{t \rightarrow \infty} p(t) = p_+ \quad \text{and} \quad \lim_{t \rightarrow -\infty} p(t) = p_-$$

and according to Proposition 2.6, the equation $x' = Df(p(t))x$ has the same exponential dichotomy than $x' = Df(p_+)x$ (resp. $x' = Df(p_-)x$) on $[t_0, \infty)$ (resp. $(-\infty, t_0]$). That is, if the stable (resp. unstable) subspace of $x' = Df(p_+)x$ (resp. $x' = Df(p_-)x$) has dimension k then $x' = Df(p(t))x$ has an exponential dichotomy on $[t_0, \infty)$ (resp. $(-\infty, t_0]$) with stable subspace $E_{t_0}^s$ (resp. unstable subspace $E_{t_0}^u$) with dimension k . In fact we have the following result:

Proposition 2.7. *Let $p(t)$ be a solution of the equation $x' = f(x)$ parametrizing an orbit on the stable (resp. unstable) manifold of an equilibrium point p . Hence the variational equation $x' = Df(p(t))x$ has exponential dichotomy on $[t_0, \infty)$ (resp. $(-\infty, t_0]$). Moreover,*

$$\mathcal{R}(\mathcal{P}(t_0)) = T_{p(t_0)}W^s(p) \quad (\text{resp. } \mathcal{N}(\mathcal{P}(t_0)) = T_{p(t_0)}W^u(p)).$$

Now we can apply to (2.3) the result below, which relates the existence of bounded solutions for a linear equation and for its adjoint.

Theorem 2.8. [43, Lemma 4.2] *Let $A(t)$ be a bounded and continuous matrix defined on $(-\infty, \infty)$. The linear equation $x' = A(t)x$ has exponential dichotomy on $[t_0, \infty)$ and on $(-\infty, t_0]$ if and only if the linear operator*

$$L : x(t) \in C_b^1(\mathbb{R}, \mathbb{R}^n) \mapsto x'(t) - A(t)x(t) \in C_b^0(\mathbb{R}, \mathbb{R}^n)$$

is Fredholm. The index of L is $\dim E_{t_0}^s + \dim E_{t_0}^u - n$. Moreover, $b \in \mathcal{R}(L)$ if and only if

$$\int_{-\infty}^{\infty} \langle w(t), b(t) \rangle dt = 0$$

*for all bounded solutions $w(t)$ of the adjoint equation $w' = -A(t)^*w$.*

To explore the existence of bounded solutions of the adjoint equation one has to study its properties of exponential dichotomy.

2.2. Exponential dichotomy for the adjoint equation. Let $X(t)$ be a fundamental matrix of the equation $x' = A(t)x$. It is well known that the conjugate transpose of its inverse $X^{-1}(t)^*$ is a fundamental matrix of the adjoint equation $w' = -A(t)^*w$. From this relationship between the fundamental matrices of both equations we can conclude the following result about the connection between their respective dichotomies.

Proposition 2.9. *If the linear equation $x' = A(t)x$ has exponential dichotomy on J with projection matrix $\mathcal{P}(t)$ then the adjoint equation $w' = -A(t)^*w$ has exponential dichotomy on J with projection matrix $I - \mathcal{P}(t)^*$. Moreover, for each $t_0 \in J$*

$$\begin{aligned} \mathbb{R}^n &= \mathcal{R}(\mathcal{P}(t_0)) \perp \mathcal{R}(I - \mathcal{P}(t_0)^*) = \mathcal{R}(\mathcal{P}(t_0)) \perp \mathcal{N}(\mathcal{P}(t_0)^*), \\ \mathbb{R}^n &= \mathcal{R}(I - \mathcal{P}(t_0)) \perp \mathcal{R}(\mathcal{P}(t_0)^*) = \mathcal{N}(\mathcal{P}(t_0)) \perp \mathcal{R}(\mathcal{P}(t_0)^*). \end{aligned}$$

As done in Definition 2.3 we can define now the stable and unstable subspaces for adjoint equations.

Definition 2.10. *Suppose that $J = [\tau, \infty)$ (resp. $J = (-\infty, \tau]$) is contained in the interval of definition of $x' = A(t)x$. For each $t_0 \in J$ the stable (resp. unstable) subspace for initial time $t = t_0$ of the adjoint equation $x' = -A(t)^*x$ is defined as*

$$\begin{aligned} E_{t_0}^{s*} &= \{w \in \mathbb{R}^n : \|X^{-1}(t)^*X(t_0)^*w\| \rightarrow 0 \text{ when } t \rightarrow \infty\} \\ (\text{resp. } E_{t_0}^{u*} &= \{w \in \mathbb{R}^n : \|X^{-1}(t)^*X(t_0)^*w\| \rightarrow 0 \text{ when } t \rightarrow -\infty\}). \end{aligned}$$

The following result about the relationship between the invariant subspaces of the equation $x' = A(t)x$ and its adjoint follows as a straight consequence of Proposition 2.4 and Proposition 2.9.

Proposition 2.11. *Suppose that the equation $x' = A(t)x$ with $x \in \mathbb{R}^n$ and $t \in J$ has exponential dichotomy in J .*

(1) *If $J = [t_0, \infty)$ then*

$$E_{t_0}^s = \mathcal{R}(\mathcal{P}(t_0)) = \{x \in \mathbb{R}^n : \sup_{t \geq t_0} \|X(t)X^{-1}(t_0)x\| < \infty\},$$

$$E_{t_0}^{s*} = \mathcal{N}(\mathcal{P}(t_0)^*) = \{w \in \mathbb{R}^n : \sup_{t \geq t_0} \|X^{-1}(t)^*X(t_0)^*w\| < \infty\},$$

$$\text{and } \mathbb{R}^n = E_{t_0}^s \perp E_{t_0}^{s*}.$$

(2) *If $J = (-\infty, t_0]$ then*

$$E_{t_0}^u = \mathcal{N}(\mathcal{P}(t_0)) = \{x \in \mathbb{R}^n : \sup_{t \leq t_0} \|X(t)X^{-1}(t_0)x\| < \infty\},$$

$$E_{t_0}^{u*} = \mathcal{R}(\mathcal{P}(t_0)^*) = \{w \in \mathbb{R}^n : \sup_{t \leq t_0} \|X^{-1}(t)^*X(t_0)^*w\| < \infty\},$$

$$\text{and } \mathbb{R}^n = E_{t_0}^u \perp E_{t_0}^{u*}.$$

In short, if the linear equation $x' = A(t)x$ has exponential dichotomy in $J = [t_0, \infty)$ (resp. $(-\infty, t_0]$) then the forward (resp. backward) bounded solutions of this equation and its adjoint are those which tend to zero exponentially when $t \rightarrow \infty$ (resp. $t \rightarrow -\infty$). On the other hand, from the decompositions of \mathbb{R}^n given in Proposition 2.11 it follows that, if $x' = A(t)x$ has m linearly independent forward (resp. backward) bounded solutions, then the adjoint equation $w' = -A(t)^*w$ has $n - m$ linearly independent forward (resp. backward) bounded solutions.

Proposition 2.12. *If the linear equation $x' = A(t)x$ has exponential dichotomy in $[t_0, \infty)$ and in $(-\infty, t_0]$ then the number of linearly independent bounded solutions of the adjoint equation $w' = -A(t)^*w$ is*

$$\dim E_{t_0}^{s*} \cap E_{t_0}^{u*} = n - \dim E_{t_0}^s - \dim E_{t_0}^u + \dim E_{t_0}^s \cap E_{t_0}^u.$$

Now we apply the above result to determine the number of bounded solutions of the adjoint equation $z' = -Df(p(t))^*z$. As we have already noticed, the number of linearly independent forward (resp. backward) bounded solutions of the variational equation $x' = Df(p(t))x$ is given by the dimension of the stable (resp. unstable) subspace of the equation $x' = Df(p_+)x$ (resp. $x' = Df(p_-)x$). That is, such number coincides with the dimension of $W^s(p_+)$ (resp. $W^u(p_-)$). Therefore, taking into account that $E_{t_0}^s = T_{p(t_0)}W^s(p_+)$ and $E_{t_0}^u = T_{p(t_0)}W^u(p_-)$, we can conclude, from Proposition 2.12, the following result.

Proposition 2.13. *If $p(t)$ is a (homo)heteroclinic solution connecting two equilibrium points p_+ and p_- then the number of linearly independent bounded solutions of the adjoint variational equation $w' = -Df(p(t))^*w$ is the codimension of $T_{p(t_0)}W^s(p_+) + T_{p(t_0)}W^u(p_-)$, that is,*

$$n - \dim W^s(p_+) - \dim W^u(p_-) + \dim T_{p(t_0)}W^s(p_+) \cap T_{p(t_0)}W^u(p_-).$$

Definition 2.14. A (homo)heteroclinic orbit γ is said non degenerate if

$$\dim T_p W^s(p_+) \cap T_p W^u(p_-) = 1,$$

with $p \in \gamma$. Otherwise γ is said degenerate.

Remark 2.15. If the (homo)heteroclinic orbit is non degenerate, the number of linearly independent bounded solutions is obtained directly from the stability indexes of p_+ and p_- . Moreover, although $\dim T_{p(t_0)} W^s(p_+) = \dim W^s(p_+)$ and $\dim T_{p(t_0)} W^u(p_-) = \dim W^u(p_-)$, in general $\dim T_{p(t_0)} W^s(p_+) \cap T_{p(t_0)} W^u(p_-)$ does not coincide with $\dim W^s(p_+) \cap W^u(p_-)$.

In the sequel the (homo)heteroclinic orbit $\gamma = \{p(t) : t \in \mathbb{R}\}$ will be non degenerate.

2.3. Bifurcation equation. As already mentioned, the existence of (homo)heteroclinic orbits for (2.1) implies the existence of bounded solutions of (2.2) and, consequently, the existence of bounded solutions of (2.3) when $b(t) \in C_b^0(\mathbb{R}, \mathbb{R}^n)$. According to Proposition 2.8, if the adjoint variational equation $w' = -Df(p(t))^* w$ has d linearly independent bounded solutions w_i , then the persistence of the (homo)heteroclinic orbit requires the fulfillment of the d conditions

$$\int_{-\infty}^{\infty} \langle w_i(t), b(t) \rangle dt = 0 \quad \text{for } i = 1, \dots, d.$$

The question now is the sufficiency of such conditions.

When $d = 1$ the sufficiency could be followed from [10]. In general, for $d \geq 1$, the techniques to be used follow the first steps of the Lin's method [34, 50]. For $\|\lambda\|$ small enough, one has to look for solutions $p_\lambda^+(\cdot)$ and $p_\lambda^-(\cdot)$ of (2.1), contained in the stable and unstable invariant manifolds of the equilibrium points $p_+(\lambda)$ and $p_-(\lambda)$, respectively (see Figure 1). Initial values $p_\lambda^\pm(t_0)$ will belong to a section Σ_{t_0} transverse to the (homo)heteroclinic orbit γ . Namely

$$\Sigma_{t_0} = p(t_0) + \{f(p(t_0))\}^\perp = p(t_0) + (W_{t_0}^+ \oplus W_{t_0}^- \oplus E_{t_0}^*)$$

where $E_{t_0}^* = E_{t_0}^{s*} \cap E_{t_0}^{u*}$ and $W_{t_0}^+$ (resp. $W_{t_0}^-$) is the orthogonal complement of

$$E_{t_0}^s \cap E_{t_0}^u = \text{span}\{f(p(t_0))\} \text{ in } E_{t_0}^s \text{ (resp. } E_{t_0}^u).$$

Moreover the condition $\xi^\infty(\lambda) = p_\lambda^-(t_0) - p_\lambda^+(t_0) \in E_{t_0}^*$ will be required. Under these assumptions there will exist two unique solutions $p_\lambda^\pm(\cdot)$ for each λ . The jump

$$\xi^\infty(\lambda) = p_\lambda^-(t_0) - p_\lambda^+(t_0)$$

measures the displacement between the stable and unstable invariant manifolds on the section Σ_{t_0} in the direction of the subspace $E_{t_0}^* = [E_{t_0}^s + E_{t_0}^u]^\perp$.

The proof of the result below can be found in [50, Lemma 3.3] and [30, Lemma 2.1.2]. Namely, in [30] only the first item is proved and, moreover, the proof is developed for the degenerate case although the non degenerate one follows in a similar manner. The second item is proved in [50] for the non degenerate case. We include in Appendix C a complete and simplified proof of this result.

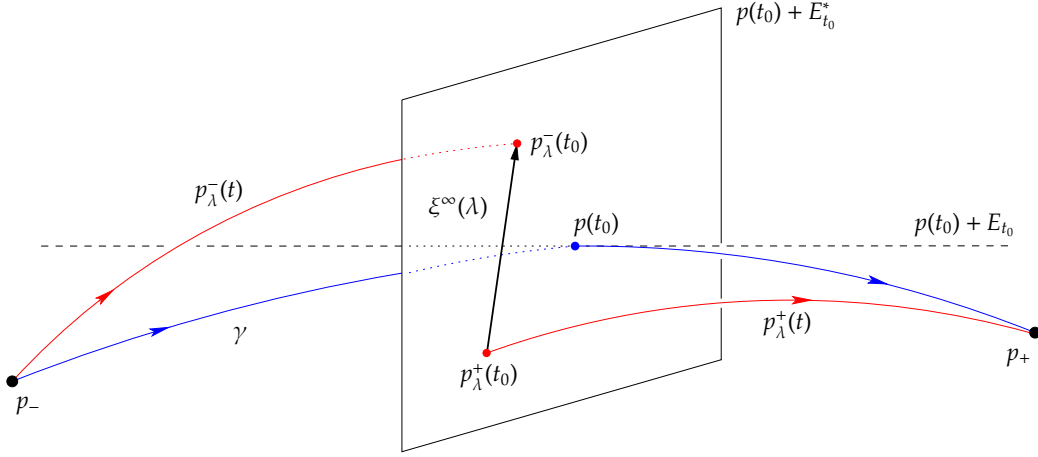


Figure 1. Non-degenerate heteroclinic orbit in \mathbb{R}^3 where the 1-dimensional manifolds coincide. In this case, $E_{t_0}^s = E_{t_0}^u = E_{t_0}$ (unidimensional), $E_{t_0}^{s*} = E_{t_0}^{u*} = E_{t_0}^*$ (bidimensional) and $\Sigma_{t_0} = p(t_0) + E_{t_0}^*$. For simplicity, we have assume that the perturbation satisfies $g(\lambda, p_{\pm}) = 0$ for all λ .

Lemma 2.16. *There exists $\delta > 0$ such that for all $\lambda \in \mathbb{R}^k$, with $\|\lambda\| < \delta$,*

- (1) *There exists a unique pair of solutions $p_{\lambda}^+(t)$ and $p_{\lambda}^-(t)$ of (2.1) parametrizing orbits on $W^s(p_+(\lambda))$ and $W^u(p_-(\lambda))$, respectively, such that $p_{\lambda}^{\pm}(t_0) \in \Sigma_{t_0}$ and*

$$\xi^{\infty}(\lambda) = p_{\lambda}^-(t_0) - p_{\lambda}^+(t_0) \in E_{t_0}^*.$$

Writing the solutions as $p_{\lambda}^{\pm}(t) = p(t) + z_{\lambda}^{\pm}(t)$, then $z_{\lambda}^{\pm}(\cdot)$ are, respectively, forward and backward bounded solutions of the equation (2.2). They depend regularly on λ and the functions z_0^{\pm} are identically zero.

- (2) *For $\varepsilon > 0$ small enough, there exists a (homo)heteroclinic solution $p_{\lambda}(t)$ such that*

$$\|p_{\lambda}(t_0) - p(t_0)\| < \varepsilon \text{ if and only if } \xi^{\infty}(\lambda) = 0,$$

that is, the components $\xi_i^{\infty}(\lambda)$ of $\xi^{\infty}(\lambda)$ in a basis $\{w_i : i = 1 \dots d\}$ of $E_{t_0}^$ satisfy*

$$\xi_i^{\infty}(\lambda) \equiv \int_{-\infty}^{t_0} \langle w_i(s), b(\lambda, s, z_{\lambda}^-(s)) \rangle ds + \int_{t_0}^{\infty} \langle w_i(s), b(\lambda, s, z_{\lambda}^+(s)) \rangle ds = 0$$

being $w_i(s) = X^{-1}(s)^ X(t_0)^* w_i$ for $i = 1, \dots, d$ bounded linearly independent solutions of the adjoint variational equation.*

According with the above statement the persistence of (homo)heteroclinic orbits follows from the analysis of the bifurcation equation $\xi^{\infty}(\lambda) = 0$. The existence of non zero parameter values $\lambda \in \mathbb{R}^k$ such that $\xi^{\infty}(\lambda) = 0$ follows from the Implicit Function Theorem when $D_{\lambda} \xi^{\infty}(0)$ has rank $d < k$. Thus, the following result follows:

Theorem 2.17. Let $\xi^\infty(\lambda) = 0$, with $\lambda \in \mathbb{R}^k$, be the bifurcation equation of the differential equation (2.1). If $k > d$ and $\text{rank } D_\lambda \xi^\infty(0) = d$, then (2.1) has a (homo)heteroclinic orbit for each parameter value λ on a regular manifold of dimension $k - d$ with tangent subspace at $\lambda = 0$ given by the solutions of the system

$$\sum_{j=1}^k \xi_{ij}^\infty \lambda_j = 0 \quad i = 1, \dots, d$$

where

$$\xi_{ij}^\infty \equiv \frac{\partial \xi_i^\infty}{\partial \lambda_j}(0) = \int_{-\infty}^{\infty} \langle w_i(s), D_{\lambda_j} g(0, p(s)) \rangle ds$$

for $i = 1, \dots, d$ and $j = 1, \dots, k$.

Remark 2.18. Note that, when $k \leq d$, $\lambda = 0$ is the unique value of $\lambda \in \mathbb{R}^k$ for which there exists a (homo)heteroclinic orbit $\gamma_\lambda = \{p_\lambda(t) : p'_\lambda(t) = f(p_\lambda(t)) + g(\lambda, p_\lambda(t)) \text{ } t \in \mathbb{R}\}$ such that $\sup_{t \in \mathbb{R}} \|p_\lambda(t) - p(t)\|$ is small enough. If $k > d$ the homoclinic connection persists for parameter values on a manifold of codimension d where

$$d = n - \dim W^s(p_+) - \dim W^u(p_-) + 1.$$

In such a case we say that there is (homo)heteroclinic bifurcation of a non degenerate orbit at $\lambda = 0$ which is of codimension d .

3. NILPOTENT SINGULARITIES OF CODIMENSION n ON \mathbb{R}^n

3.1. Generic unfoldings. Let X be a C^∞ vector field in \mathbb{R}^n with $X(0) = 0$ and 1-jet at the origin linearly conjugated to $\sum_{k=1}^{n-1} x_{k+1} \partial / \partial x_k$. Introducing appropriate C^∞ coordinates, X can be written as:

$$\sum_{k=1}^{n-1} x_{k+1} \frac{\partial}{\partial x_k} + f(x_1, \dots, x_n) \frac{\partial}{\partial x_n}, \quad (3.1)$$

with $f(x) = O(\|x\|^2)$ where $x = (x_1, \dots, x_n)$. It is said that X has a nilpotent singularity of codimension n at 0 if the generic condition $\partial^2 f / \partial x_1^2(0) \neq 0$ is fulfilled. The vector field X itself will be often referred to as a nilpotent singularity of codimension n .

Nilpotent singularities of codimension n are generic in families depending on at least n parameters and according with [18, Lemma 2.1] we can state the following result:

Lemma 3.1. Any n -parametric generic unfolding of a nilpotent singularity of codimension n in \mathbb{R}^n can be written as

$$\sum_{k=1}^{n-1} x_{k+1} \frac{\partial}{\partial x_k} + \left(\mu_1 + \sum_{k=2}^n \mu_k x_k + x_1^2 + h(x, \mu) \right) \frac{\partial}{\partial x_n}, \quad (3.2)$$

where $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n$, $h(0, \mu) = 0$, $(\partial h / \partial x_i)(0, \mu) = 0$ for $i = 1, \dots, n$, $(\partial^2 h / \partial x_1^2)(0, \mu) = 0$, $h(x, \mu) = O(\|(x, \mu)\|^2)$ and $h(x, \mu) = O(\|(x_2, \dots, x_n)\|)$.

Remark 3.2. Besides the condition $\partial^2 f / \partial x_1^2(0) \neq 0$ in (3.1), genericity assumptions in Lemma 3.1 include a transversality condition involving derivatives of the family with respect to parameters.

The classical techniques of reduction to normal forms could be used to remove terms in the Taylor expansion of h but we do not need to work with simpler expressions. To obtain the results provided in the next sections we will have to impose

$$\kappa = \frac{\partial^2 h}{\partial x_1 \partial x_2}(0, 0) \neq 0, \quad (3.3)$$

as an additional generic assumption.

3.2. Rescalings and limit families. Generalizing the techniques used in [14] for dimension three, we rescale variables and parameters by means of

$$\begin{aligned} \mu_1 &= \varepsilon^{2n} v_1, \\ \mu_k &= \varepsilon^{n-k+1} v_k \quad \text{for } k = 2, \dots, n, \\ x_k &= \varepsilon^{n+k-1} y_k \quad \text{for } k = 1, \dots, n, \end{aligned} \quad (3.4)$$

with $\varepsilon > 0$ and $v_1^2 + \dots + v_n^2 = 1$, and also multiply the whole family by a factor $1/\varepsilon$. In new coordinates and parameters (3.2) can be written as

$$\sum_{k=1}^{n-1} y_{k+1} \frac{\partial}{\partial y_k} + \left(v_1 + \sum_{k=2}^n v_k y_k + y_1^2 + \varepsilon \kappa y_1 y_2 + O(\varepsilon^2) \right) \frac{\partial}{\partial y_n}, \quad (3.5)$$

with κ as introduced in (3.3) and where $y = (y_1, \dots, y_n)$ belongs to an arbitrarily big compact in \mathbb{R}^n .

The first step to understand the dynamics arising in generic unfoldings of n -dimensional nilpotent singularities of codimension n is the study of the bifurcation diagram of the *limit family*

$$\sum_{k=1}^{n-1} y_{k+1} \frac{\partial}{\partial y_k} + \left(v_1 + \sum_{k=2}^n v_k y_k + y_1^2 \right) \frac{\partial}{\partial y_n}, \quad (3.6)$$

obtained by taking $\varepsilon = 0$ in (3.5). Structurally stable behaviours and generic bifurcations in (3.6) should persist in (3.5) for $\varepsilon > 0$ small enough.

If $v_1 > 0$ then (3.6) has no equilibrium points. Moreover the function

$$L(y_1, \dots, y_n) = y_n - v_2 y_1 - v_3 y_2 - \dots - v_n y_{n-1}$$

is strictly increasing along the orbits and therefore the maximal compact invariant set is empty. Hence we only need to pay attention to the case $v_1 \leq 0$.

On the other hand, up to a change of sign, family (3.6) is invariant under the transformation

$$\begin{aligned} (v, y) &\mapsto (v_1, (-1)^{n-1} v_2, (-1)^{n-2} v_3, \dots, v_{n-1}, -v_n, \\ &\quad (-1)^n y_1, (-1)^{n-1} y_2, (-1)^{n-2} y_3, \dots, y_{n-1}, -y_n), \end{aligned} \quad (3.7)$$

with $v = (v_1, \dots, v_n)$. As a first consequence, the study of bifurcations can be reduced to the region

$$\mathcal{R} = \{(v_1, \dots, v_n) \in \mathbb{S}^{n-1} : v_1 \leq 0, v_n \leq 0\}.$$

Moreover, since the limit family is invariant under (3.7) up to a change of sign, for parameter values on the set

$$\mathcal{T} = \{(v_1, \dots, v_n) \in \mathbb{S}^{n-1} : v_{n-2i} = 0 \text{ with } i = 0, \dots, \lfloor (n-2)/2 \rfloor\},$$

where $\lfloor \cdot \rfloor$ denotes the floor function, the correspondent vector fields in the limit family (3.6) are time-reversible with respect to the involution

$$R : (y_1, y_2, y_3, \dots, y_n) \mapsto ((-1)^n y_1, (-1)^{n-1} y_2, \dots, y_{n-1}, -y_n).$$

We said that the manifold \mathcal{T} of dimension $\lfloor n/2 \rfloor - 1$ is the *reversibility set* of the n -dimensional nilpotent limit family.

Note that the divergence of the limit family (3.6) takes the constant value v_n . Therefore the condition $v_n = 0$ characterizes a subfamily of volume-preserving vector fields. Assuming that n is even and defining $m = n/2$, for parameter values on the reversibility set the limit family (3.6) can be written as

$$\sum_{k=1}^{n-1} y_{k+1} \frac{\partial}{\partial y_k} + \left(v_1 + \sum_{k=1}^{m-1} v_{2k+1} y_{2k+1} + y_1^2 \right) \frac{\partial}{\partial y_n}. \quad (3.8)$$

In Appendix A we will prove the following result.

Theorem 3.3. *Introducing the new variables $q = S \cdot (y_1, y_3, \dots, y_{n-1})^t$ and $p = (y_2, y_4, \dots, y_n)^t$, with*

$$S = \begin{pmatrix} -v_3 & -v_5 & \dots & -v_{n-1} & 1 \\ -v_5 & & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -v_{n-1} & \ddots & \ddots & & \vdots \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix},$$

the family (3.8) transforms into

$$\frac{\partial H}{\partial p} \frac{\partial}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial}{\partial p},$$

where

$$H(q, p) = \frac{1}{2} \langle Sp, p \rangle + V(q).$$

The potential V is defined as

$$\begin{aligned} V(q) = & -\frac{1}{3} q_m^3 - \frac{1}{2} \sum_{k=1}^{m-1} v_{2k+1} b_{k+1} q_m^2 - \frac{1}{2} \sum_{j=1}^{\lfloor m/2 \rfloor} b_{m-2j+1} q_{m-j}^2 \\ & - \sum_{k=1}^{m-1} \sum_{i=m-k}^{m-1} v_{2k+1} b_{i-m+k+1} q_i q_m - \sum_{j=1}^{\lfloor m/2 \rfloor} \sum_{i=j}^{m-j-1} b_i q_i q_{m-j} - v_1 q_m, \end{aligned}$$

where, given $b_1 = 1$,

$$b_i = \sum_{\ell=1}^{i-1} v_{2(m-i+\ell)+1} b_\ell \quad \text{for } i = 2, \dots, m.$$

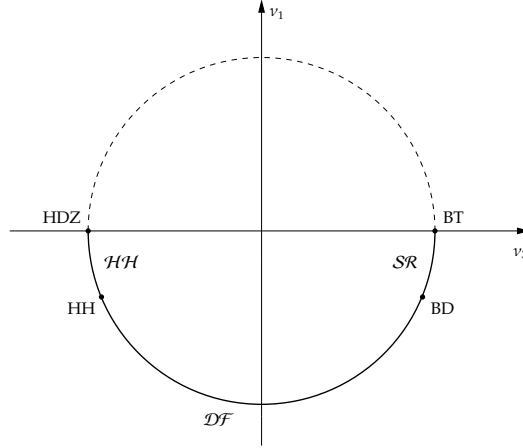


Figure 2. The reversibility curve \mathcal{T} is split into several arcs attending to the type of eigenvalues of the linear part of (4.2) at p_- .

4. NILPOTENT SINGULARITY OF CODIMENSION 4 IN \mathbb{R}^4 .

We will prove that in any generic unfolding of a nilpotent singularity of codimension four in \mathbb{R}^4 there exists a bifurcation hypersurface of homoclinic connections to bifocus equilibria.

Along this section we will take $n = 4$ in all the general expressions introduced in §3. It follows from Lemma 3.1 that any generic unfolding of the nilpotent singularity of codimension four in \mathbb{R}^4 can be written as in (3.2). After applying the rescaling (3.4) we get

$$y_2 \frac{\partial}{\partial y_1} + y_3 \frac{\partial}{\partial y_2} + y_4 \frac{\partial}{\partial y_3} + (v_1 + v_2 y_2 + v_3 y_3 + v_4 y_4 + y_1^2 + \varepsilon \kappa y_1 y_2 + O(\varepsilon^2)) \frac{\partial}{\partial y_4}, \quad (4.1)$$

with $v = (v_1, v_2, v_3, v_4) \in \mathbb{S}^3$ and $\varepsilon > 0$.

As mentioned in §3.2 the first step to understand the dynamics arising in (4.1) is the study of the limit family

$$y_2 \frac{\partial}{\partial y_1} + y_3 \frac{\partial}{\partial y_2} + y_4 \frac{\partial}{\partial y_3} + (v_1 + v_2 y_2 + v_3 y_3 + v_4 y_4 + y_1^2) \frac{\partial}{\partial y_4}, \quad (4.2)$$

obtained from (4.1) taking $\varepsilon = 0$. As argued in §3.2 one only need to pay attention to parameters in the region $\mathcal{R} = \{(v_1, v_2, v_3, v_4) \in \mathbb{S}^3 : v_1 \leq 0, v_4 \leq 0\}$. When $v \in \mathcal{R}$, vector fields in the limit family (4.2) have equilibrium points $p_{\pm} = (\pm \sqrt{-v_1}, 0, 0, 0)$ with characteristic equations

$$r^4 - v_4 r^3 - v_3 r^2 - v_2 r \mp 2 \sqrt{-v_1} = 0. \quad (4.3)$$

Local bifurcations arising in the family were discussed in [19].

For parameters on the reversibility curve $\mathcal{T} = \{(v_1, v_2, v_3, v_4) \in \mathbb{S}^3 : v_2 = v_4 = 0\}$ with $v_1 \leq 0$, the characteristic equations reduces to $r^4 - v_3 r^2 \mp 2 \sqrt{-v_1} = 0$. It follows that the linear part at p_+ always have a pair of real eigenvalues and a pair of complex eigenvalues with non-zero

real part. Local behaviour at p_- is richer and it is depicted in Figure 2. Note that we only have to pay attention to $v_1^2 + v_3^2 = 1$ with $v_1 \leq 0$. It easily follows that the linear part at p_- has

- a double zero eigenvalue and eigenvalues ± 1 at $BT = (0, 0, 1, 0)$,
- a double zero eigenvalue and a pair of pure imaginary eigenvalues at $HDZ = (0, 0, -1, 0)$,
- two double real eigenvalues $\pm(v_3/2)^{1/2}$ at $BD = (v_1, 0, v_3, 0)$ with $v_3^2 - 8\sqrt{-v_1} = 0$ and $v_3 > 0$,
- two double pure imaginary eigenvalues $\pm i(-v_3/2)^{1/2}$ at $HH = (v_1, 0, v_3, 0)$ with $v_3^2 - 8\sqrt{-v_1} = 0$ and $v_3 < 0$,
- four non-zero real eigenvalues $\pm\lambda_k$, with $k = 1, 2$ for parameters along the open arc \mathcal{SR} between BD and BT ,
- four complex eigenvalues with non-zero real part $\rho \pm \omega i$ and $-\rho \pm \omega i$ for parameters along the open arc \mathcal{DF} between BD and HH ,
- four pure imaginary eigenvalues $\pm\omega_k i$, with $k = 1, 2$, for parameters along the open arc \mathcal{HH} between HH and HDZ .

From the analysis of the linear part at the equilibrium points it follows that a bifocus is only possible at p_- . In order to show the existence of bifocal homoclinic bifurcations in the unfolding of the nilpotent singularity of codimension four in \mathbb{R}^4 we must study the existence of homoclinic orbits to p_- for parameter values along \mathcal{T} .

To study the family (4.1) close to the reversibility curve \mathcal{T} with $v_1 < 0$ it is more convenient to use a directional version of the rescaling (3.4) taking $v_1 = -1$ and $(v_2, v_3, v_4) = (\bar{v}_2, \bar{v}_3, \bar{v}_4) \in \mathbb{R}^3$ to get

$$y_2 \frac{\partial}{\partial y_1} + y_3 \frac{\partial}{\partial y_2} + y_4 \frac{\partial}{\partial y_3} + \left(-1 + \bar{v}_2 y_2 + \bar{v}_3 y_3 + \bar{v}_4 y_4 + y_1^2 + \varepsilon \kappa y_1 y_2 + O(\varepsilon^2) \right) \frac{\partial}{\partial y_4}. \quad (4.4)$$

The equilibrium points when $\varepsilon = 0$ are given by $q_{\pm} = (\pm 1, 0, 0, 0)$. Note that in fact q_{\pm} are the only equilibrium points even for $\varepsilon > 0$ because in (3.2) $h(x, \mu) = O(\|(x_2, \dots, x_n)\|)$ and this property is preserved by the rescaling. In order to compare with equations already considered in the literature we translate q_- to the origin applying the change of coordinates

$$x_1 = (y_1 + 1)/2, \quad x_2 = y_2/2^{5/4}, \quad x_3 = y_3/2^{6/4}, \quad x_4 = y_4/2^{7/4},$$

to (4.4) and multiplying by the factor $2^{1/4}$ to obtain

$$x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} + \left(-x_1 + \eta_2 x_2 + \eta_3 x_3 + \eta_4 x_4 + x_1^2 + \bar{\varepsilon} \kappa x_1 x_2 + O(\bar{\varepsilon}^2) \right) \frac{\partial}{\partial x_4} \quad (4.5)$$

with $\eta_2 = 2^{-3/4}(\bar{v}_2 - \varepsilon \kappa)$, $\eta_3 = 2^{-1/2}\bar{v}_3$, $\eta_4 = 2^{-1/4}\bar{v}_4$ and $\bar{\varepsilon} = 2^{1/4}\varepsilon$. The equilibrium point q_- in (4.4) corresponds to the equilibrium point of (4.5) at the origin. The limit subfamily for $\eta_2 = \eta_4 = \bar{\varepsilon} = 0$ is now given as

$$x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} + \left(-x_1 + \eta_3 x_3 + x_1^2 \right) \frac{\partial}{\partial x_4}. \quad (4.6)$$

Writing $u = x_1$, (4.6) is equivalent to the fourth order differential equation

$$u^{(iv)}(t) + Pu''(t) + u(t) - u(t)^2 = 0, \quad (4.7)$$

with $P = -\eta_3$. As already mentioned in the introduction, the above equation has been extensively studied in the literature.

In [1] authors prove that (4.7) can be written as a hamiltonian system (as we have stated in Theorem 3.3 for a more general case) satisfying the hypothesis required in [24, Theorem 2] to conclude that, for each $P \leq -2$, there exists an even solution u with $u(t) \rightarrow 0$ when $t \rightarrow \pm\infty$ satisfying that $u > 0$, $u' < 0$ and $(P/2)u' + u''' > 0$ on $(0, \infty)$. They also prove that for all $P \leq -2$ any such even solution is unique. From [3] it follows that this unique homoclinic orbit is transversal for the restriction to the level surface of the hamiltonian function which contains it and, consequently, it is non degenerate in the sense of Definition 2.14. Moreover, again in [1], the persistence of such homoclinic solutions is argued for $P > -2$ but close enough to -2 . Variational methods used in [8] allow to prove that at least one homoclinic solution exists for $P < 2$. On the other hand, in [3, Section 2] authors check all hypothesis required in [12, Theorem 4.4] to conclude that a Belyakov-Devaney bifurcation takes place at $P = -2$. It consists in the emerging from the primary homoclinic solution and for each $n \in \mathbb{N}$ of a finite number of n -modal secondary homoclinics (or n -pulses) which cut n times a section transversal to the primary homoclinic orbit [13, 5, 7]. Heuristic arguments in [3], supported by numerical results, show that the non-degenerate n -modal homoclinic orbits arising at $P = -2$ become in degenerate orbits and disappear gradually when P varies from $P = -2$ to $P = 2$ through a cascade of coalescences and bifurcations. In particular, it is known from [27] that for P close to $P = 2$ there exist at least two even homoclinic solutions and from the numerical results it seems that no other homoclinic orbits reaches $P = 2$.

All the above results about the existence of homoclinic solutions of (4.7) can be directly translated to family (4.6) and also to the reversible subfamily of (4.2) obtained restricting to parameter values along the previously defined reversibility curve \mathcal{T} . For the later case we can conclude that (see Figure 2)

- for parameter values along $\mathcal{DF} \cup \{\text{BD}\} \cup \mathcal{SR}$ there exists a symmetric homoclinic orbit at p_- which is unique and non degenerate along $\{\text{BD}\} \cup \mathcal{SR}$,
- BD is a Belyakov-Devaney bifurcation point,
- numerical continuation shows that the non degenerate n -modal homoclinic orbits arising at BD become in degenerate orbits and disappear gradually when parameters move along \mathcal{DF} in the direction of HH. Close to that point only two symmetric homoclinic orbits persist.

To study the persistence of homoclinic orbits we will consider (4.1) as an unfolding of the Belyakov-Devaney bifurcation point BD. As already mentioned it is better to work with expression (4.5) for the rescaled unfolding. With respect to parameters $(\eta_2, \eta_3, \eta_4, \bar{\epsilon})$ the point BD corresponds to $(0, 2, 0, 0)$. Note that (4.5) can be written as

$$x' = f(x) + g(\lambda, x), \quad (4.8)$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\eta_2, \eta_3 - 2, \eta_4, \bar{\varepsilon})$,

$$f(x) = (x_2, x_3, x_4, -x_1 + 2x_3 + x_1^2)$$

and

$$g(\lambda, x) = (0, 0, 0, \lambda_1 x_2 + \lambda_2 x_3 + \lambda_3 x_4 + \lambda_4 \kappa x_1 x_2 + O(\lambda_4^2)).$$

As already mentioned, q_{\pm} are the only equilibrium points of (4.4) for all $\varepsilon \geq 0$ and hence $g(\lambda, 0) = 0$ for all λ . Observe that only bifurcations occurring inside the region of parameters with $\lambda_4 > 0$ will be observed in the unfolding of the singularity. Family (4.8) fulfills all the hypothesis imposed to (2.1). In particular, $x' = f(x)$ satisfies the following:

(BD1): It has a first integral

$$H(x_1, x_2, x_3, x_4) = \frac{1}{2}x_1^2 - \frac{1}{3}x_1^3 - x_2^2 + x_2x_4 - \frac{1}{2}x_3^2$$

(BD2): It is time reversible with respect to

$$R : (x_1, x_2, x_3, x_4) \mapsto (x_1, -x_2, x_3, -x_4).$$

(BD3): The origin is a hyperbolic equilibrium point at which the linear part has a pair of double real eigenvalues ± 1 .

(BD4): According to [1], there exists a non degenerate homoclinic orbit $\gamma = \{p(t) = (p_1(t), p_2(t), p_3(t), p_4(t)) : t \in \mathbb{R}\}$ to the origin such that $p_1(t)$ and $p_3(t)$ are even functions and $p_2(t)$ and $p_4(t)$ are odd functions and, moreover, $p_1 > 0$, $p_2 < 0$ and $p_4 - p_2 > 0$ on $(0, \infty)$.

(BD5): According to Proposition 2.13, since γ is non degenerate, both the variational equation $z' = Df(p(t))z$ and its adjoint $z' = -Df(p(t))^*z$ has a unique non trivial linearly independent bounded solution. The function $\varphi(t) = f(p(t))$ is a bounded solution of the variational equation and

$$\psi(t) = \nabla H(p(t)) = (p_1(t) - p_1(t)^2, p_4(t) - 2p_2(t), -p_3(t), p_2(t))$$

is a bounded solution of adjoint equation.

Finally, let us consider the bifurcation equation for homoclinic solutions $\xi^\infty(\lambda) = 0$, with $\xi^\infty : \Lambda \rightarrow \mathbb{R}$ and $\Lambda \subset \mathbb{R}^4$ a neighbourhood of the origin, as introduced in Lemma 2.16. It follows from Theorem 2.17 that under the generic condition

$$\nabla \xi^\infty(0) = (\xi_{\lambda_1}, \xi_{\lambda_2}, \xi_{\lambda_3}, \xi_{\lambda_4}) \neq 0$$

where

$$\xi_{\lambda_i} = \int_{-\infty}^{\infty} \langle \psi(t), \frac{\partial g}{\partial \lambda_i}(0, p(t)) \rangle dt,$$

then (4.8) has homoclinic orbits (continuation of γ) for parameters on a hypersurface $\mathcal{H}om$ with tangent subspace at $\lambda = 0$ given by

$$\xi_{\lambda_1} \lambda_1 + \xi_{\lambda_2} \lambda_2 + \xi_{\lambda_3} \lambda_3 + \xi_{\lambda_4} \lambda_4 = 0. \quad (4.9)$$

Note that

$$\begin{aligned}\xi_{\lambda_1} &= \int_{-\infty}^{\infty} p_2^2(t) dt, & \xi_{\lambda_2} &= \int_{-\infty}^{\infty} p_2(t)p_3(t) dt, \\ \xi_{\lambda_3} &= \int_{-\infty}^{\infty} p_2(t)p_4(t) dt, & \xi_{\lambda_4} &= \int_{-\infty}^{\infty} \kappa p_1(t)p_2^2(t) dt.\end{aligned}$$

Clearly $\xi_{\lambda_1} \neq 0$. Since p_2p_3 is an odd function $\xi_{\lambda_2} = 0$. Integrating by parts one gets $\xi_{\lambda_3} = -\int_{-\infty}^{\infty} p_3(t)^2 dt \neq 0$. Finally, since p_1 is a positive function, we also get that $\xi_{\lambda_4} \neq 0$. Therefore the tangent subspace (4.9) intersects $\lambda_4 = 0$ transversely. Consequently $\mathcal{H}om$ also meets $\lambda_4 = 0$ transversely.

Now we have to study the eigenvalues at the equilibrium point in order to determine which types of homoclinic orbits can be unfolded by the singularity. Since for $\lambda = 0$ the linear part at $x = 0$ has a pair of double real eigenvalues ± 1 and $\dim W^s(0) = \dim W^u(0) = 2$, for all λ small enough, then we can expect three different types of equilibrium: a focus-focus (bifocus), a node-node or a focus-node. It easily follows that the characteristic polynomial at $x = 0$ is given by

$$Q(r, \lambda) = r^4 - D(\lambda)r^3 - C(\lambda)r^2 - B(\lambda)r - A(\lambda),$$

with

$$\begin{aligned}A(\lambda) &= -1 + O(\lambda_4^2) & B(\lambda) &= \lambda_1 + O(\lambda_4^2) \\ C(\lambda) &= 2 + \lambda_2 + O(\lambda_4^2) & D(\lambda) &= \lambda_3 + O(\lambda_4^2).\end{aligned}$$

The condition for an improper node is given by the discriminant equations

$$Q(r, \lambda) = 0, \quad \frac{\partial Q}{\partial r}(r, \lambda) = 0.$$

Note that $(r, \lambda) = (\pm 1, 0)$ are both solutions of the discriminant equations. Now it follows from a straightforward application of the Implicit Function Theorem that there exist two hypersurfaces \mathcal{D}^- and \mathcal{D}^+ through the origin in the parameter space such that, for parameter values on \mathcal{D}^- (resp. \mathcal{D}^+) the equilibrium point at the origin has a double negative (resp. positive) real eigenvalue. Moreover the respective tangent subspaces at $\lambda = 0$ are $\lambda_1 - \lambda_2 + \lambda_3 = 0$ and $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Let $N_{\mathcal{H}om} = (\xi_{\lambda_1}, \xi_{\lambda_2}, \xi_{\lambda_3}, \xi_{\lambda_4})$, $N_{\mathcal{D}^-} = (1, -1, 1, 0)$ and $N_{\mathcal{D}^+} = (1, 1, 1, 0)$ be the normal vectors to the tangent spaces of $\mathcal{H}om$, \mathcal{D}^- and \mathcal{D}^+ at $\lambda = 0$, respectively. Moreover denote $N_{\lambda_4=0} = (0, 0, 0, 1)$. Since $\text{rank}(N_{\mathcal{H}om}, N_{\mathcal{D}^-}, N_{\lambda_4=0}) = 3$, there exists a surface $\mathcal{H}om^- = \mathcal{H}om \cap \mathcal{D}^-$ transverse to $\lambda_4 = 0$ of homoclinic orbits to an equilibrium point with a double negative real eigenvalue. Moreover, since $\text{rank}(N_{\mathcal{H}om}, N_{\mathcal{D}^+}, N_{\lambda_4=0}) = 3$, there exists a surface $\mathcal{H}om^+ = \mathcal{H}om \cap \mathcal{D}^+$ transverse to $\lambda_4 = 0$ of homoclinic orbits to an equilibrium point with a double positive real eigenvalue. On the other hand $\text{rank}(N_{\mathcal{H}om}, N_{\mathcal{D}^-}, N_{\mathcal{D}^+}, N_{\lambda_4=0}) = 4$ if and only if $\xi_{\lambda_1} - \xi_{\lambda_3} \neq 0$. But, taking into account that p_2 and p_4 are odd functions and also that $p_2 < 0$ and $p_4 - p_2 > 0$ on $(0, \infty)$ it follows that

$$\xi_{\lambda_1} - \xi_{\lambda_3} = \int_{-\infty}^{\infty} p_2(t)(p_2(t) - p_4(t)) dt > 0.$$

Hence we can conclude that $\text{rank}(N_{\mathcal{H}om}, N_{\mathcal{D}^-}, N_{\mathcal{D}^+}, N_{\lambda_4=0}) = 4$. Therefore there exists a curve $\mathcal{H}om^\pm = \mathcal{H}om \cap \mathcal{D}^- \cap \mathcal{D}^+$ transverse to $\lambda_4 = 0$ of homoclinic orbits to an equilibrium point with a pair of double real eigenvalues one positive and the other negative.

Summarizing, we have proved the following result

Theorem 4.1. *In a neighbourhood of $\lambda = 0$ there exists a bifurcation hypersurface $\mathcal{H}om$ corresponding to parameter values for which (4.8) has homoclinic orbits to the origin. Moreover there exist two bifurcation surfaces $\mathcal{H}om^-$ and $\mathcal{H}om^+$ contained in $\mathcal{H}om$ corresponding to parameter values for which the origin has a double negative and positive, respectively, real eigenvalue. The surfaces $\mathcal{H}om^+$ and $\mathcal{H}om^-$ intersect transversely along a curve $\mathcal{H}om^\pm$ corresponding to parameter values for which the origin has a pair of double real eigenvalues $\{r_1, r_2\}$ with $r_1 < 0 < r_2$. $\mathcal{H}om^- \cup \mathcal{H}om^+$ splits $\mathcal{H}om$ into four regions:*

- (i) $\mathcal{H}om_{FF}$: homoclinic orbits to a focus-focus equilibrium (bifocus case),
- (ii) $\mathcal{H}om_{N^+F^-}$: homoclinic orbits to a (repelling) node-(attracting) focus equilibrium,
- (iii) $\mathcal{H}om_{F^+N^-}$: homoclinic orbits to a (repelling) focus-(attracting) node equilibrium,
- (iv) $\mathcal{H}om_{NN}$: homoclinic orbits to a node-node equilibrium.

All bifurcations are transverse to $\lambda_4 = 0$.

Since all bifurcations are transverse to $\lambda_4 = 0$ they are also present in the unfolding of the nilpotent singularity of codimension four. Particularly, Theorem B follows as a corollary of Theorem 4.1.

Remark 4.2. *Recall that the bifurcation point $\lambda = 0$ in (4.8) corresponds to the Belyakov-Devaney bifurcation point BD in (4.1). In particular, the hypersurface of parameters corresponding to homoclinic orbits to a node-node equilibrium point in family (4.1) cuts $\varepsilon = 0$ along the curve \mathcal{SR} . This follows by restricting the bifurcation analysis to $\lambda_4 = 0$.*

5. NILPOTENT SINGULARITY OF CODIMENSION 3 IN \mathbb{R}^3 .

We will prove that in any generic unfolding of the nilpotent singularity of codimension three in \mathbb{R}^3 there exists a one-side bifurcation curve of topological Bykov cycles.

Along this section we will take $n = 3$ in all the general expressions introduced in §3. It follows from Lemma 3.1 that any generic unfolding of the nilpotent singularity of codimension three in \mathbb{R}^3 can be written as in (3.2). After applying the rescaling (3.4) we get

$$y_2 \frac{\partial}{\partial y_1} + y_3 \frac{\partial}{\partial y_2} + \left(v_1 + v_2 y_2 + v_3 y_3 + y_1^2 + \varepsilon \kappa y_1 y_2 + O(\varepsilon^2) \right) \frac{\partial}{\partial y_3}. \quad (5.1)$$

with $v = (v_1, v_2, v_3) \in \mathbb{S}^2$ and $\varepsilon > 0$.

As mentioned in §3.2 the first step to understand the dynamics arising in (5.1) is the study of the limit family

$$y_2 \frac{\partial}{\partial y_1} + y_3 \frac{\partial}{\partial y_2} + \left(v_1 + v_2 y_2 + v_3 y_3 + y_1^2 \right) \frac{\partial}{\partial y_3}.$$

The above family has already been treated in the literature and discussions about several aspects of the dynamics can be seen in [14, 15, 16, 29] and references there included. As in the case of the nilpotent singularity of codimension four in \mathbb{R}^4 we are interested in the

dynamics close to the reversibility curve $\mathcal{T} = \{(v_1, v_2, v_3) \in \mathbb{S}^2 : v_3 = 0\}$. Particularly, we will pay attention to parameters with $v_1 < 0$ and $v_2 < 0$.

To study family (5.1) close to the reversibility curve it is more convenient to use a directional version of the rescaling (3.4) taking $v_2 = -1$ and $(v_1, v_3) = (\bar{v}_1, \bar{v}_3) \in \mathbb{R}^2$ to get

$$y_2 \frac{\partial}{\partial y_1} + y_3 \frac{\partial}{\partial y_2} + (\bar{v}_1 - y_2 + \bar{v}_3 y_3 + y_1^2 + \varepsilon \kappa y_1 y_2 + O(\varepsilon^2)) \frac{\partial}{\partial y_3}.$$

Moreover, in order to use results already present in the literature, we introduce new variables $(x_1, x_2, x_3) = -2(y_1, y_2, y_3)$ and write $-2\bar{v}_1 = c^2$ when $v_1 < 0$ to obtain

$$x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + \left(c^2 - x_2 + \bar{v}_3 x_3 - \frac{1}{2} x_1^2 - 2\varepsilon \kappa x_1 x_2 + O(\varepsilon^2) \right) \frac{\partial}{\partial x_3}. \quad (5.2)$$

In the above expression, taking the limit case $\varepsilon = 0$ and also $\bar{v}_3 = 0$ we get the 1-parameter family

$$x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + \left(c^2 - x_2 - \frac{1}{2} x_1^2 \right) \frac{\partial}{\partial x_3}.$$

As mentioned in the introduction the above family has been extensively studied in the literature. It is commonly referred as the *Michelson system* and it has the following properties:

- (M1): For all $c \geq 0$ the system has only two equilibrium points $Q_{\pm} = (\pm \sqrt{2}c, 0, 0)$ where the characteristic equation of the linear part is given by $r^3 + r \mp \sqrt{2}c$.
- (M2): The eigenvalues at Q_+ (resp. Q_-) are λ and $-\rho \pm i\omega$ (resp. $-\lambda$ and $\rho \pm i\omega$) with $\lambda > 0$, $\rho > 0$ and $\omega \neq 0$. Therefore $\dim W^s(Q_+) = \dim W^u(Q_-) = 2$. Moreover, since the Michelson system has zero divergence, $\lambda - 2\rho = 0$ and hence $\rho < \lambda$.
- (M3): It is time-reversible with respect to the involution

$$R : (x_1, x_2, x_3) \mapsto (-x_1, x_2, -x_3).$$

- (M4): It follows from [31] that the Michelson system has a solution $p(t)$ given by

$$p_1(t) = \alpha(-9 \tanh \beta t + 11 \tanh^3 \beta t), \quad p_2(t) = p'_1(t), \quad p_3(t) = p''_1(t) \quad (5.3)$$

with $\alpha = 15 \sqrt{11/19^3}$ and $\beta = \sqrt{11/19}/2$, when $c = c_k = \sqrt{2}\alpha$. It parametrizes an orbit Γ_1 along which two branches of the 1-dimensional invariant manifolds coincide. The orbit Γ_1 is invariant by the reversibility, that is,

$$p(-t) = (p_1(-t), p_2(-t), p_3(-t)) = Rp(t) = (-p_1(t), p_2(t), -p_3(t)).$$

Hence p_1 and p_3 are odd functions and p_2 is an even function.

- (M5): It follows from [29] that when $c = c_k$ there also exists an orbit $\Gamma_2 = W^s(Q_+) \cap W^u(Q_-)$. Moreover the intersection is topologically transversal.

Putting together (M4) and (M5) it follows that

- (M6): When $c = c_k$ the Michelson system has a topological Bykov cycle.

Let us recall the notion of topological Bykov cycle.

Definition 5.1. Consider a vector field in \mathbb{R}^3 with two hyperbolic equilibrium points p_{\pm} satisfying $\dim W^s(p_+) = \dim W^u(p_-) = 2$. Any heteroclinic cycle consisting of two heteroclinic orbits $\Gamma_1 \subseteq W^u(p_+) \cap W^s(p_-)$ and $\Gamma_2 \subseteq W^s(p_+) \cap W^u(p_-)$ between the equilibrium points p_{\pm} such that the intersection along Γ_2 is transversal is called a T-point. When at p_{\pm} the linear part has complex eigenvalues the T-point is called a Bykov cycle. In both cases, if the intersection along Γ_2 is only topologically transversal we refer to a topological T-point or a topological Bykov cycle.

Remark 5.2. Let us recall the notion of topological transversality. Let D be a 2-dimensional open disk transversal to the flow at some point $T \in \Gamma_2$. Let S and U be the connected components of $W^u(P_-) \cap D$ and $W^s(P_+) \cap D$, respectively, containing the point T . If the diameter of D is small enough both S and U split D into two connected components. The connection Γ_2 is said topologically transversal if the two connected components of $S \setminus \{T\}$ belong to different connected components of $D \setminus U$.

To study the persistence of the Bykov cycle we will consider (5.2) as an unfolding of the Michelson system at the Kuramoto point $c = c_k$ introduced in (M4). Note that (5.2) can be written as

$$x' = f(x) + g(\lambda, x), \quad (5.4)$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3) = (c^2 - c_k^2, \bar{v}_3, \varepsilon)$,

$$f(x) = (x_2, x_3, c_k^2 - x_2 - \frac{1}{2}x_1^2)$$

and

$$g(\lambda, x) = (0, 0, \lambda_1 + \lambda_2 x_3 - 2\lambda_3 x_1 x_2 + O(\lambda_3^2)).$$

Bifurcations occurring inside the region of parameters with $\lambda_3 > 0$ will be observed in the unfolding of the singularity. Family (5.4) fulfills all the hypothesis imposed to (2.1). Particularly $x' = f(x)$ satisfies all properties from (M1) to (M6).

The heteroclinic orbit Γ_1 is non degenerate of codimension two in the sense of Definition 2.14. Hence the variational equation $z'(t) = Df(p(t))z(t)$ has a unique (up to multiplicative constants) bounded solution $f(p(t))$ whereas the adjoint variational equation $w'(t) = -Df(p(t))^*w(t)$ has a pair of linearly independent bounded solutions, again according to Proposition 2.13. Let $\varphi(t) = (\varphi_1(t), \varphi_2(t), \varphi_3(t))$ and $\psi(t) = (\psi_1(t), \psi_2(t), \psi_3(t))$ two of such solutions. Since $\varphi(t) \wedge \psi(t)$ is a bounded solution of the variational equation it follows that the plane determined by $\varphi(t)$ and $\psi(t)$ is orthogonal to $f(p(t))$ for all values of t . Therefore, all solutions of the adjoint variational equations with initial conditions on $f(p(0))^{\perp}$ are bounded solutions.

On the other hand, it easily follows that $w'(t) = -Df(p(t))^*w(t)$ is invariant under the involutions $(w_1, w_2, w_3) \mapsto (-w_1, w_2, -w_3)$ and $(w_1, w_2, w_3) \mapsto (w_1, -w_2, w_3)$ and the time reverse $t \mapsto -t$. Therefore, taking $\varphi(0) = (0, -1, 0)$ and $\psi(0) = (1 - c_k^2/p_2(0), 0, 1)$ we can conclude that $\varphi(t)$ and $\psi(t)$ are bounded solutions of the adjoint variational equation and that φ_1, φ_3 and ψ_2 are odd functions whereas φ_2, ψ_1 and ψ_3 are even functions.

Consider now the bifurcation equation for homoclinic solutions $\xi^{\infty}(\lambda) = 0$, with $\xi^{\infty} = (\xi_1, \xi_2) : \Lambda \rightarrow \mathbb{R}^2$ and $\Lambda \subset \mathbb{R}^3$ a neighbourhood of the origin, as introduced in Lemma 2.16.

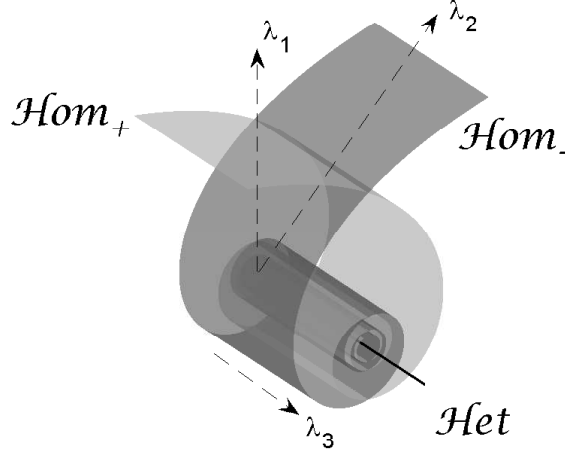


Figure 3. Sketch showing the bifurcation curve $\mathcal{H}et$ to topological Bykov cycles in family (5.4) and the spirals of bifurcation to homoclinic orbits $\mathcal{H}om_+$ and $\mathcal{H}om_-$ for Q_+ and Q_- , respectively. Note that when $\lambda_3 = 0$, the existence of $\mathcal{H}om_-$ follows from the symmetry with respect to (3.7).

It follows from Theorem 2.17 that, under the generic condition $\text{rank} D_\lambda \xi^\infty(0) = 2$, where

$$D_\lambda \xi^\infty(0) = \begin{pmatrix} \xi_{1,\lambda_1} & \xi_{1,\lambda_2} & \xi_{1,\lambda_3} \\ \xi_{2,\lambda_1} & \xi_{2,\lambda_2} & \xi_{2,\lambda_3} \end{pmatrix}$$

with

$$\begin{aligned} \xi_{1,\lambda_1} &= \int_{-\infty}^{\infty} \varphi_3(t) dt, & \xi_{2,\lambda_1} &= \int_{-\infty}^{\infty} \psi_3(t) dt, \\ \xi_{1,\lambda_2} &= \int_{-\infty}^{\infty} \varphi_3(t) p_3(t) dt, & \xi_{2,\lambda_2} &= \int_{-\infty}^{\infty} \psi_3(t) p_3(t) dt, \\ \xi_{1,\lambda_3} &= \int_{-\infty}^{\infty} -2\kappa \varphi_3(t) p_1(t) p_2(t) dt, & \xi_{2,\lambda_3} &= \int_{-\infty}^{\infty} -2\kappa \psi_3(t) p_1(t) p_2(t) dt, \end{aligned} \quad (5.5)$$

then (5.4) has heteroclinic orbits (continuation of Γ_1) for parameters on a bifurcation curve $\mathcal{H}et$ with tangent subspace at $\lambda = 0$ given by the intersection of the planes

$$\begin{aligned} \xi_{1,\lambda_1} \lambda_1 + \xi_{1,\lambda_2} \lambda_2 + \xi_{1,\lambda_3} \lambda_3 &= 0 \\ \xi_{2,\lambda_1} \lambda_1 + \xi_{2,\lambda_2} \lambda_2 + \xi_{2,\lambda_3} \lambda_3 &= 0. \end{aligned}$$

From the parities of p_1, p_2, p_3, φ_3 and ψ_3 it follows that

$$\xi_{1,\lambda_1} = \xi_{2,\lambda_2} = \xi_{2,\lambda_3} = 0.$$

On the other hand, in Appendix B we will show that

$$\xi_{1,\lambda_2} \neq 0, \quad \xi_{1,\lambda_3} \neq 0, \quad \xi_{2,\lambda_1} \neq 0.$$

Hence, it is a straightforward application of the Implicit Function Theorem that, indeed, there exists a bifurcation curve $\mathcal{H}et$ of heteroclinic connections along the one dimensional

invariant manifolds. Moreover it easily follows that the tangent space at $\lambda = 0$ is generated by a vector $(0, -\xi_{1,\lambda_3}/\xi_{1,\lambda_2}, 1)$ and hence $\mathcal{H}et$ intersects $\lambda_3 = 0$ transversely. Since Γ_2 is a topologically transverse intersection, $\mathcal{H}et$ is a bifurcation curve of topological Bykov cycles. This concludes the proof of Theorem C which was stated in the introduction.

As already mentioned there is a Shil'nikov bifurcation surface $\mathcal{H}om_+$ shaped as a scroll around $\mathcal{H}et$ (see Figure 3) corresponding to parameter values for which the system has a Shil'nikov homoclinic orbit to Q_+ . Note that the Shil'nikov condition is open and hence it follows from (M2). Moreover since the trace of the linear part at Q_+ is given by λ_2 the dissipative condition is also satisfied in $\mathcal{H}om_+ \cap \{\lambda \in \mathbb{R}^3 : \lambda_2 < 0\}$. Hence strange attractors exist for parameter values on positive Lebesgue measure set.

Remark 5.3. In [16] the existence of subsidiary Bykov cycles in the Michelson system for values of c close to c_k is also discussed. Moreover, in [35] the accumulation of Bykov cycles when $c \rightarrow 0$ is also argued. In all cases the heteroclinic connections have the symmetry properties that we have just used. It should be possible to extend our result to conclude that in (5.2), and consequently in (5.1), there exist more bifurcation curves to Bykov cycles and particularly an infinite sequence of such type of bifurcation curves. Nevertheless, the generic conditions on the bifurcation equation need to be checked.

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APPENDIX A. PROOF OF THEOREM 3.3

We will use the following technical result:

Lemma A.1. *Given a symmetric upper anti-triangular matrix*

$$A = \begin{pmatrix} a_m & a_{m-1} & \dots & a_2 & 1 \\ a_{m-1} & & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_2 & \ddots & \ddots & & \vdots \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix},$$

A^{-1} is a lower anti-triangular symmetric matrix

$$A^{-1} = \begin{pmatrix} 0 & \dots & \dots & 0 & 1 \\ \vdots & & \ddots & \ddots & b_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & & b_{m-1} \\ 1 & b_2 & \dots & b_{m-1} & b_m \end{pmatrix}$$

where, given $b_1 = 1$,

$$b_i = - \sum_{\ell=1}^{i-1} a_{i-\ell+1} b_\ell$$

for $i = 2, \dots, m$.

Proof. Let P be an anti-diagonal matrix with all entries equal to 1. Hence

$$L = PA = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ a_2 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{m-1} & & \ddots & \ddots & 0 \\ a_m & a_{m-1} & \dots & a_2 & 1 \end{pmatrix}.$$

is a lower triangular matrix. Therefore, $L^{-1} = (b_{i,j})$ is also a lower triangular matrix and hence $A^{-1} = L^{-1}P^{-1} = L^{-1}P$ is a lower anti-triangular matrix. In fact, using the well know formulas for the calculation of the inverse of a triangular matrix, it follows that, for all $j = 1, \dots, m$

$$\begin{aligned} b_{j,j} &= 1, \\ b_{i,j} &= 0 \quad \text{for all } i = 1, \dots, j-1, \\ b_{i,j} &= - \sum_{\ell=j}^{i-1} a_{i-\ell+1} b_{\ell,j} \quad \text{for all } i = j+1, \dots, m. \end{aligned}$$

On the other hand, $b_{i,j} = b_{i+1,j+1}$ for all $i = j + 1, \dots, m - 1$. Indeed it is clear for $i = j + 1$. For $i = j + 2, \dots, m - 1$ we can argue by induction.

Finally, by defining $b_i = b_{i,1}$ for all $i = 1, \dots, m$, and calculating $A^{-1} = L^{-1}P$ the proof is finished. \square

It follows from Lemma A.1 that

$$S^{-1} = \begin{pmatrix} 0 & \dots & \dots & 0 & 1 \\ \vdots & & \ddots & \ddots & b_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & & b_{m-1} \\ 1 & b_2 & \dots & b_{m-1} & b_m \end{pmatrix},$$

where, defining $b_1 = 1$,

$$b_i = \sum_{\ell=1}^{i-1} v_{2(m-i+\ell)+1} b_\ell \quad \text{for } i = 2, \dots, m.$$

or equivalently

$$b_{m-j+1} = \sum_{\ell=1}^{m-j} v_{2(j+\ell-1)+1} b_\ell = \sum_{k=j}^{m-1} v_{2k+1} b_{k-j+1} \quad \text{for } j = 1, \dots, m-1. \quad (\text{A.1})$$

Writing family (3.8) in the new variables we get

$$Sp \frac{\partial}{\partial q} + \sum_{k=1}^{m-1} \left(\sum_{i=m-k}^m b_{i-m+k+1} q_i \right) \frac{\partial}{\partial p_k} + \left(v_1 + \sum_{k=1}^{m-1} v_{2k+1} \dot{p}_k + q_m^2 \right) \frac{\partial}{\partial p_m}.$$

To obtain a function $V(q)$ such that $\dot{p} = -\nabla V(q)$ we need $-\partial V / \partial q_i = \dot{p}_i$ for all $i = 1, \dots, m$. In particular

$$-\frac{\partial V}{\partial q_m} = v_1 + \sum_{k=1}^{m-1} v_{2k+1} \dot{p}_k + q_m^2,$$

and therefore

$$\begin{aligned} -V(q) &= v_1 q_m + \sum_{k=1}^{m-1} v_{2k+1} \left(\frac{1}{2} b_{k+1} q_m^2 + \sum_{i=m-k}^{m-1} b_{i-m+k+1} q_i q_m \right) \\ &\quad + \frac{1}{3} q_m^3 + \varphi_{m-1}(q_1, \dots, q_{m-1}). \end{aligned}$$

From the identity $-\partial V / \partial q_{m-1} = \dot{p}_{m-1}$ and taking into account the equation (A.1) we get

$$\frac{\partial \varphi_{m-1}}{\partial q_{m-1}} = \sum_{i=1}^m b_i q_i - \sum_{k=1}^{m-1} v_{2k+1} b_k q_m = \sum_{i=1}^{m-1} b_i q_i$$

and therefore

$$\varphi_{m-1}(q_1, \dots, q_{m-1}) = \frac{1}{2} b_{m-1} q_{m-1}^2 + \sum_{i=1}^{m-2} b_i q_i q_{m-1} + \varphi_{m-2}(q_1, \dots, q_{m-2}).$$

Since $-\partial V/\partial q_{m-2} = \dot{p}_{m-2}$, a similar computation leads to

$$\frac{\partial \varphi_{m-2}}{\partial q_{m-2}} = \sum_{i=2}^m b_{i-1} q_i - \sum_{k=2}^{m-1} v_{2k+1} b_{k-1} q_m - b_{m-2} q_{m-1} = \sum_{i=2}^{m-2} b_{i-1} q_i.$$

and hence

$$\varphi_{m-2}(q_1, \dots, q_{m-2}) = \frac{1}{2} b_{m-3} q_{m-2}^2 + \sum_{i=2}^{m-3} b_{i-1} q_i q_{m-2} + \varphi_{m-3}(q_1, \dots, q_{m-3}).$$

A recursive argument provides

$$\frac{\partial \varphi_{m-j}}{\partial q_{m-j}} = \sum_{i=j}^m b_{i-j+1} q_i - \sum_{k=j}^{m-1} v_{2k+1} b_{k-j+1} - \sum_{i=m-j+1}^{m-1} b_{i-j+1} q_i = \sum_{i=j}^{m-j} b_{i-j+1} q_i,$$

for all $j = 1, \dots, \lfloor m/2 \rfloor$ and consequently,

$$\begin{aligned} \varphi_{m-j}(q_1, \dots, q_{m-j}) &= \frac{1}{2} b_{m-2j+1} q_{m-j}^2 \\ &\quad + \sum_{i=j}^{m-j-1} b_i q_i q_{m-j} + \varphi_{m-j-1}(q_1, \dots, q_{m-j-1}) \end{aligned}$$

where for $j = \lfloor m/2 \rfloor$ the function $\varphi_{m-\lfloor m/2 \rfloor-1}$ is constant. Therefore we get a function $V(q)$ with

$$\begin{aligned} -V(q) &= v_1 q_m + \sum_{k=1}^{m-1} v_{2k+1} \left(\frac{1}{2} b_{k+1} q_m^2 + \sum_{i=m-k}^{m-1} b_{i-m+k+1} q_i q_m \right) + \frac{1}{3} q_m^3 \\ &\quad + \sum_{j=1}^{\lfloor m/2 \rfloor} \left(\frac{1}{2} b_{m-2j+1} q_{m-j}^2 + \sum_{i=j}^{m-j-1} b_i q_i q_{m-j} \right) + \varphi_{m-\lfloor m/2 \rfloor-1}, \end{aligned}$$

such that $\dot{p} = -\nabla V(q)$. This concludes the proof of Theorem 3.3.

APPENDIX B. ESTIMATION OF ξ_{1,λ_2} , ξ_{1,λ_3} AND ξ_{2,λ_1}

Let us write the adjoint variational equation $w'(t) = -Df(p(t))^*w(t)$ as

$$\begin{cases} w'_1(t) = p_1(t)w_3(t) \\ w'_2(t) = -w_1(t) + w_3(t) \\ w'_3(t) = -w_2(t) \end{cases} \quad (\text{B.1})$$

where $w = (w_1, w_2, w_3)$, $p_1(t) = \alpha(-9 \tanh(\beta t) + 11 \tanh^3(\beta t))$, with $\alpha = 15\sqrt{11/19^3}$ and $\beta = \sqrt{11/19}/2$, and $p = (p_1, p_2, p_3)$, with $p_2 = p'_1$ and $p_3 = p''_2$. Writing $s(t) = \tanh(\beta t)$ it follows that

$$p_2(t) = \alpha\beta(1 - s(t)^2)(-9 + 33s(t)^2) \quad \text{and} \quad p_3(t) = \alpha\beta^2(1 - s(t)^2)(84s(t) - 132s(t)^3).$$

Our goal is to show that ξ_{2,λ_1} , ξ_{1,λ_2} and ξ_{1,λ_3} , as given in (5.5), are different from zero. Note that from the parities of p_1 , p_2 , p_3 , ψ_3 and φ_3 , it follows that

$$\begin{aligned} \xi_{2,\lambda_1} &= 2 \int_0^\infty \psi_3(t) dt, \\ \xi_{1,\lambda_2} &= 2 \int_0^\infty \varphi_3(t)p_3(t) dt, \\ \xi_{1,\lambda_3} &= -4\kappa \int_0^\infty \varphi_3(t)p_1(t)p_2(t) dt. \end{aligned} \quad (\text{B.2})$$

We have already argued in §5 that each bounded solution $w(t)$ satisfies an orthogonality condition with respect to $f(p(t))$, that is,

$$p_2(t)w_1(t) + p_3(t)w_2(t) + (c_k^2 - p_2(t) - (p_1(t))^2/2)w_3(t) = 0.$$

with $c_k = \sqrt{2}\alpha$. Now, introducing $v_1(t) = w_3(t)$ and $v_2(t) = v'_1(t)$, we get equivalently

$$A(t)v'(t) = B(t)v(t), \quad (\text{B.3})$$

with $v(t) = (v_1(t), v_2(t))$ and

$$A(t) = \begin{pmatrix} 1 & 0 \\ 0 & p_2(t) \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 & 1 \\ (p_1(t))^2/2 - c_k^2 & p_3(t) \end{pmatrix}.$$

Since we have chosen bounded solutions $\varphi(t)$ and $\psi(t)$ of (B.1) satisfying the initial conditions $\varphi(0) = (0, -1, 0)$ and $\psi(0) = (1 - c_k^2/p_2(0), 0, 1)$, they correspond, with respect to the new variables v_1 and v_2 , to solutions $\hat{\varphi}(t) = (\hat{\varphi}_1(t), \hat{\varphi}_2(t))$ and $\hat{\psi}(t) = (\hat{\psi}_1(t), \hat{\psi}_2(t))$ of (B.3) with initial conditions $\hat{\varphi}(0) = (0, 1)$ and $\hat{\psi}(0) = (1, 0)$, respectively. Therefore, taking into account that $\kappa \neq 0$, we only have to show that

$$\int_0^\infty \hat{\psi}_1(t) dt \neq 0, \quad \int_0^\infty \hat{\varphi}_1(t)p_3(t) dt \neq 0 \quad \text{and} \quad \int_0^\infty \hat{\varphi}_1(t)p_1(t)p_2(t) dt \neq 0.$$

Note that $A(t)$ is singular at $\hat{t}_\pm = \tanh^{-1}(\sqrt{3/11})/\beta \approx \pm 1.5529$. Hence (B.3) must be treated as a differential algebraic equation rather than as an ordinary differential equation on the interval $[0, \infty)$. We will provide approximate values of the integrals in (B.2) using numerical methods on an interval $[0, t_0]$, with $t_0 > \hat{t}_+$ to be fixed later, and providing upper bounds for the absolute value of the integrals on $[t_0, \infty)$.

In order to get appropriate upper bounds on $[t_0, \infty)$ we write the equation (B.3) as $v'(t) = Qv(t) + R(t)v(t)$ with $R(t) = (A(t))^{-1}B(t) - Q$ and

$$Q = \lim_{t \rightarrow \infty} (A(t))^{-1}B(t) = \begin{pmatrix} 0 & 1 \\ -30/19 & -\sqrt{11/19} \end{pmatrix}.$$

The eigenvalues of Q are $\lambda_{\pm} = (-\sqrt{209} \pm i\sqrt{2071})/38$ and

$$P = \begin{pmatrix} (-\sqrt{209} - i\sqrt{2071})/60 & (-\sqrt{209} + i\sqrt{2071})/60 \\ 1 & 1 \end{pmatrix}$$

is such that $Q = PJP^{-1}$ where J is the complex canonical form of Q with entries λ_+ and λ_- along the diagonal. From the Lagrange Formula

$$v(t) = Pe^{J(t-t_0)}P^{-1}v_0 + Pe^{Jt} \int_{t_0}^t e^{-Js}P^{-1}R(s)v(s) ds,$$

for the solution $v(t)$ of (B.3) with $v(t_0) = v_0$, we obtain

$$e^{-at}\|v(t)\| \leq \|P\|e^{-at_0}\|P^{-1}\| \|v_0\| + \|P\| \int_{t_0}^t e^{-as}\|P^{-1}\| \|R(s)\| \|v(s)\| ds,$$

where a denotes the real part of λ_{\pm} . For any $\varepsilon > 0$ we can choose t_0 such such that $\|R(t)\| < \varepsilon$ for all $t \geq t_0$ and hence, applying the Gronwall Lemma we get

$$\|v(t)\| \leq \|P\| \|P^{-1}\| \|v_0\| e^{(a+\|P\| \|P^{-1}\| \varepsilon)(t-t_0)}.$$

Moreover we can assume that t_0 is large enough to have $\|R(t)\|$ strictly decreasing on $[t_0, \infty)$ and $\|P\| \|P^{-1}\| \|R(t_0)\| \ll |a|$. Therefore we can take $\varepsilon = R(t_0)$ and achieve the following upper bounds

$$\begin{aligned} \left| \int_{t_0}^{\infty} \hat{\psi}_1(t) dt \right| &\leq \int_{t_0}^{\infty} \|\hat{\psi}(t)\| dt \\ &\leq \|P\| \|P^{-1}\| \|\hat{\psi}(t_0)\| \int_{t_0}^{\infty} e^{(a+\|P\| \|P^{-1}\| \|R(t_0)\|)(t-t_0)} dt \\ &\leq L \|\hat{\psi}(t_0)\|, \end{aligned}$$

with

$$L = \frac{-\|P\| \|P^{-1}\|}{a + \|P\| \|P^{-1}\| \|R(t_0)\|},$$

and, analogously,

$$\begin{aligned} \left| \int_{t_0}^{\infty} \hat{\phi}_1(t) p_3(t) dt \right| &\leq L |p_3(t_0)| \|\hat{\phi}(t_0)\|, \\ \left| \int_{t_0}^{\infty} \hat{\phi}(t) p_1(t) p_2(t) dt \right| &\leq L |p_1(t_0) p_2(t_0)| \|\hat{\phi}(t_0)\|, \end{aligned}$$

assuming that t_0 is chosen such that $|p_3(t)|$ and $|p_1(t) p_2(t)|$ are decreasing on $[t_0, \infty)$.

To conclude we have to provide estimations of the integrals on $[0, t_0]$. One can check that, as required, the value $t_0 = 20$ is such that $\|R(t)\|$, $|p_3(t)|$ and $|p_1(t) p_2(t)|$ are decreasing on $[t_0, \infty)$. Moreover the numerical results show that the values $\|\hat{\psi}(20)\|$ and $\|\hat{\phi}(20)\|$ provide small enough upper bounds for the integrals on $[t_0, \infty)$. As already mentioned (B.3) must be treated as a differential algebraic equation. The computing environment MATLAB [36] provides codes to deal with such kind of equations. For instance, according to [51], the ode15s code is an appropriate one to solve DAEs. We have used such method working

TOL	Estimation of $\int_0^{20} \hat{\psi}_1(t) dt$	Estimation of $\int_0^{20} \hat{\phi}_1(t)p_3(t) dt$	Estimation of $\int_0^{20} \hat{\phi}_1(t)p_1(t)p_2(t) dt$
10^{-3}	-2.69317886	3.38993680	2.14437726
10^{-4}	-2.66952044	3.42255576	2.18337593
10^{-5}	-2.65457297	3.42416164	2.19168295
10^{-6}	-2.65558754	3.48165747	2.27650773
10^{-7}	-2.65577082	3.42402265	2.19092815
10^{-8}	-2.65586498	3.42412150	2.19145569
10^{-9}	-2.65592518	3.42421346	2.19175351
10^{-10}	-2.65594764	3.42423492	2.19185340
10^{-11}	-2.65596101	3.42424493	2.19188683
10^{-12}	-2.65596369	3.42424803	2.19190343
10^{-13}	-2.65596540	3.42424892	2.19190641

Table I. Estimations of the integrals on $[0, 20]$ using the code `ode15s` provided in MATLAB to deal with differential algebraic equations. Note that MATLAB uses two different tolerances, absolute and relative, to compute optimal steps. We take both equal, with the values indicated in the first column.

TOL	Estimation of $L\ \hat{\psi}(20)\ $	Estimation of $ p_3(20) L\ \hat{\phi}(20)\ $	Estimation of $ p_1(20)p_2(20) L\ \hat{\phi}(20)\ $
10^{-13}	4.219110×10^{-2}	2.103834×10^{-7}	3.321829×10^{-7}

Table II. Upper bounds of the integrals on the interval $[20, \infty]$.

with different tolerances to get the results shown in Table I. Once the numerical solution is obtained, the integrals were approximated using the trapezoidal rule with the nodes which were generated by the numerical algorithm. With the highest tolerance value, the maximum step sizes are 0.01355215 and 0.01323550 for the solutions $\hat{\psi}$ and $\hat{\phi}$, respectively. On the other hand one can get accurate estimations of the upper bounds for the integrals on $[t_0, \infty]$. The results taking the approximate values of the solutions at $t = t_0$ obtained with the highest tolerance are shown in Table II.

Remark B.1. All the estimations of upper bounds are obtained using the maximum norm. It can be easily checked that $\|P\| = 2$, $\|P^{-1}\| = \sqrt{30/109} + 30/\sqrt{2071}$ and $\|R(t)\| = (|p_1(t)^2/2 - c_k^2 + 30/19| + |p_3(t) + \sqrt{11/19}|)/|p_2(t)|$.

Remark B.2. The code `ode23t` included in MATLAB also solves differential algebraic equations. Even `ode45` can be used to solve our particular equation. The results does not change significantly, namely, the differences are below 10^{-4} . We have also used our own algorithms (a Taylor method of order 25 for the variational equation with small fixed step and projection on the stable manifold). Again the differences are below 10^{-4} .

APPENDIX C. PROOF OF LEMMA 2.16

As it stated in Proposition 2.7 the variational equation $z' = Df(p(t))z$ has exponential dichotomy in $[t_0, \infty)$ and $(-\infty, t_0]$. Let

$$\mathcal{P}_+(t_0) = X(t_0)P_+X^{-1}(t_0) \quad \text{and} \quad I - \mathcal{P}_-(t_0) = I - X(t_0)P_-X^{-1}(t_0)$$

be the corresponding projection matrix on the stable space $E_{t_0}^s = T_{p(t_0)}W^s(p_+)$ and instable space $E_{t_0}^u = T_{p(t_0)}W^u(p_-)$, respectively.

Before to give the proof of Lemma 2.16 we need the following preliminar result.

Lemma C.1. *Let $b \in C_b^0([t_0, \infty), \mathbb{R}^n)$. Then, $z^+(t)$ is a positively bounded solution of*

$$z' = Df(p(t))z + b(t)$$

if and only if

$$\begin{aligned} z^+(t) &= X(t)X^{-1}(t_0)\mathcal{P}_+(t_0)z^+(t_0) \\ &\quad + \int_{t_0}^t X(t)X^{-1}(s)\mathcal{P}_+(s)b(s)ds - \int_t^\infty X(t)X^{-1}(s)(I - \mathcal{P}_+(s))b(s)ds. \end{aligned} \quad (\text{C.1})$$

Proof. Since $X(t)$ is the fundamental matrix of the lineal homogeneous equation $z' = Df(p(t))z$, the solution of the complete linear equation $z' = Df(p(t))z + b(t)$ are

$$z(t) = X(t)X^{-1}(t_0)z(t_0) + X(t) \int_{t_0}^t X^{-1}(s)b(s)ds.$$

By means of the projection P_+ in the exponential dichotomy of the homogeneous equation in $[t_0, \infty)$, this solution can be written as

$$\begin{aligned} z(t) &= X(t)P_+X^{-1}(t_0)z(t_0) + X(t)(I - P_+)X^{-1}(t_0)z(t_0) \\ &\quad + X(t) \int_{t_0}^t P_+X^{-1}(s)b(s)ds + X(t) \int_{t_0}^t (I - P_+)X^{-1}(s)b(s)ds. \end{aligned} \quad (\text{C.2})$$

On the other hand, according to the exponential dichotomy $\|X(t)P_+X^{-1}(s)\| \leq Ke^{-\alpha(t-s)}$ for $t \geq s \geq t_0$, it follows that

$$\begin{aligned} |X(t)P_+X^{-1}(t_0)z(t_0)| &\leq Ke^{-\alpha(t-t_0)}|z(t_0)| \quad \text{for } t \geq t_0, \\ |X(t) \int_{t_0}^t P_+X^{-1}(s)b(s)ds| &\leq \int_{t_0}^t Ke^{-\alpha(t-s)}|b(s)|ds \quad \text{for } t \geq t_0, \end{aligned}$$

and thus, the first and third term of (C.2) are bounded for $t \geq t_0$.

If we assume that $z(t)$ is bounded, necessarily then the sum

$$\begin{aligned} &X(t)(I - P_+)X^{-1}(t_0)z(t_0) + X(t) \int_{t_0}^t (I - P_+)X^{-1}(s)b(s)ds \\ &= X(t)[(I - P_+)X^{-1}(t_0)z(t_0) + \int_{t_0}^t (I - P_+)X^{-1}(s)b(s)ds] \end{aligned} \quad (\text{C.3})$$

is also bounded. However, the exponential dichotomy $\|X(t)(I - P_+)X^{-1}(s)\| \leq Le^{-\beta(s-t)}$ for $s \geq t \geq t_0$, implies that

$$\begin{aligned} |X(t_0)(I - P_+)X^{-1}(t_0)z(t_0)| &= |X(t_0)(I - P_+)X^{-1}(t)X(t)(I - P_+)X^{-1}(t_0)z(t_0)| \\ &\leq Le^{-\beta(t-t_0)}|X(t)(I - P_+)X^{-1}(t_0)z(t_0)|. \end{aligned}$$

Hence,

$$|X(t)(I - P_+)X^{-1}(t_0)z(t_0)| \geq |X(t_0)(I - P_+)X^{-1}(t_0)z(t_0)|L^{-1}e^{\beta(t-t_0)}$$

is not bounded and since $|X(t)(I - P_+)X^{-1}(t_0)z(t_0)| \leq \|X(t)\|(I - P_+)X^{-1}(t_0)z(t_0)$, it follows that the matrix $X(t)$ is not bounded. Therefore, from (C.3) we get that the solution $z(t)$ only can be bounded if it holds

$$(I - P_+)X^{-1}(t_0)z(t_0) + \int_{t_0}^{\infty} (I - P_+)X^{-1}(s)b(s) ds = 0.$$

Consequently, from (C.2) we obtain that if $z^+(t)$ is a bounded solution of $z' = Df(p(t))z + b(t)$ then

$$\begin{aligned} z^+(t) &= X(t)X^{-1}(t_0)\mathcal{P}_+(t_0)z^+(t_0) \\ &\quad + \int_{t_0}^t X(t)X^{-1}(s)\mathcal{P}_+(s)b(s) ds - \int_t^{\infty} X(t)X^{-1}(t_0)(I - \mathcal{P}_+(s))b(s) ds. \end{aligned}$$

Conversely, to verify that $z^+(t)$ given in (C.1) is a bounded solution of $z' = Df(p(t))z + b(t)$ it suffices to see that

$$L^+b(t) = \int_{t_0}^t X(t)X^{-1}(s)\mathcal{P}_+(s)b(s) ds - \int_t^{\infty} X(t)X^{-1}(t_0)(I - \mathcal{P}_+(s))b(s) ds$$

is a particular bounded solution of the above complete linear equation. Indeed,

$$\begin{aligned} \frac{d}{dt}L^+b(t) &= Df(p(t))X(t)\left[\int_{t_0}^t P_+X^{-1}(s)b(s) ds - \int_t^{\infty} (I - P_+)X^{-1}(s)b(s) ds\right] \\ &\quad + X(t)\left[P_+X^{-1}(t)b(t) + (I - P_+)X^{-1}(t)b(t)\right] = Df(p(t))L^+b(t) + b(t), \\ |L^+b(t)| &\leq \int_{t_0}^t Ke^{-\alpha(t-s)}|b(s)| ds + \int_t^{\infty} Le^{-\beta(s-t)}|b(s)| ds \leq \tilde{K} + \tilde{L}, \end{aligned}$$

for all $t \geq t_0$, where the constant \tilde{K} and \tilde{L} not depend on t . \square

In the same way, a similar result to the negative bounded solutions of the complete linear equation is followed.

Lemma C.2. *Let $b \in C_a^0((-\infty, t_0], \mathbb{R}^n)$. Then, $z^-(t)$ is a negative bounded solution of*

$$z' = Df(p(t))z + b(t)$$

if and only if

$$\begin{aligned} z^-(t) &= X(t)X^{-1}(t_0)(I - \mathcal{P}_-(t_0))z^-(t_0) \\ &\quad + \int_{-\infty}^t X(t)X^{-1}(s)\mathcal{P}_-(s)b(s) ds - \int_t^{t_0} X(t)X^{-1}(s)(I - \mathcal{P}_-(s))b(s) ds. \end{aligned} \tag{C.4}$$

Remark C.3. Notice that the functions given in (C.1) and (C.4) are both solutions of the equation $z' = Df(p(t))z + b(t)$ for any continuous function $b(t)$ on $[t_0, \infty)$ and $(-\infty, t_0]$, respectively. On the other hand, to prove that both solutions are bounded solutions of this equation we need to use that $b(t)$ is also bounded.

Recall that $E_{t_0}^s = T_{p(t_0)}W^s(p_+)$ and $E_{t_0}^u = T_{p(t_0)}W^u(p_-)$. Notice that since the (homo)heteroclinic orbit γ is non degenerate, $E_{t_0} = E_{t_0}^s \cap E_{t_0}^u$ is one-dimensional. In fact, this space is generated by the vector $p'(t_0) = f(p(t_0))$. Moreover, according to Remark 2.15 the dimension of

$$E_{t_0}^* = E_{t_0}^{s*} \cap E_{t_0}^{u*} = [E_{t_0}^s + E_{t_0}^u]^\perp$$

is $d = s_- - s_+ + 1$ where s_\pm are the stable indexes of p_+ and p_- respectively.

Let us define $W_{t_0}^\pm$ the ortogonal complement to E_{t_0} in the tangent space of the stable and unstable manifolds, respectively, that is $E_{t_0}^s = E_{t_0} \oplus W_{t_0}^+$ and $E_{t_0}^u = E_{t_0} \oplus W_{t_0}^-$. Then

$$\mathbb{R}^n = \text{span}\{f(p(t_0))\} \oplus W_{t_0}^+ \oplus W_{t_0}^- \oplus E_{t_0}^*. \quad (\text{C.5})$$

Finally, we take the transversal section to γ at $p(t_0)$ given by

$$\Sigma_{t_0} = p(t_0) + \{f(p(t_0))\}^\perp = p(t_0) + [W_{t_0}^+ \oplus W_{t_0}^- \oplus E_{t_0}^*]$$

and a base $\{w_i : i = 1 \dots d\}$ de $E_{t_0}^*$. Notice that

$$w_i(s) = X^{-1}(s)^* X(t_0)^* w_i \quad \text{for } i = 1, \dots, d$$

are bounded linearly independent solutions of the adjoint variational equation.

Proof of Lemma 2.16. The solutions $p_\lambda(t)$ of (2.1) can be written as $p_\lambda(t) = p(t) + z_\lambda(t)$ where $z_\lambda(t)$ is a solution of (2.2). Let $Y_{t_0} = W_{t_0}^+ \oplus W_{t_0}^- \oplus E_{t_0}^*$. Assuming $p_\lambda(t_0) \in \Sigma_{t_0}$ then $z_\lambda(t_0) \in Y_{t_0}$.

In order to get that $p_\lambda(t)$ parametrizes an orbit in $W^s(p_+(\lambda))$ (resp. $W^u(p_-(\lambda))$), the function $z_\lambda(t)$ have to be a positively (resp. negatively) bounded solution of (2.2). Now if we assume that $z^\pm(t)$ are a pair of positively and negatively bounded solutions of (2.2) respectively, then $b(\cdot, z^\pm(\cdot), \lambda) \in C_b^0(J_\pm, \mathbb{R}^n)$ where $J_+ = [t_0, \infty)$ and $J_- = (-\infty, t_0]$. Thus, according to Lemma C.1 and Lemma C.2 it must be met that

$$\begin{aligned} z^+(t) &= X(t)X^{-1}(t_0)\mathcal{P}_+(t_0)z^+(t_0) \\ &+ \int_{t_0}^t X(t)X^{-1}(s)\mathcal{P}_+(s)b(s, z^\pm(s), \lambda) ds - \int_t^\infty X(t)X^{-1}(s)(I - \mathcal{P}_+(s))b(s, z^\pm(s), \lambda) ds \end{aligned} \quad (\text{C.6})$$

and

$$\begin{aligned} z^-(t) &= X(t)X^{-1}(t_0)(I - \mathcal{P}_-(t_0))z^-(t_0) \\ &+ \int_{-\infty}^t X(t)X^{-1}(s)\mathcal{P}_-(s)b(s, z^\pm(s), \lambda) ds - \int_t^{t_0} X(t)X^{-1}(s)(I - \mathcal{P}_-(s))b(s, z^\pm(s), \lambda) ds. \end{aligned} \quad (\text{C.7})$$

Conversely, according to Remark C.3, the solutions $z^+(t)$ and $z^-(t)$ of the integral equations (C.6) and (C.7) are both solutions of (2.2), but not necessarily bounded. The existence of positively and negatively bounded solutions of (C.6) and (C.7) will be proved as an application of Implicit Function Theorem.

Ler $\eta^+ = \mathcal{P}_+(t_0)z^+(t_0) \in W_{t_0}^+$ and $\eta^- = (I - \mathcal{P}_-(t_0))z^-(t_0) \in W_{t_0}^-$. Equations (C.6) and (C.7) can be written in the form

$$z^\pm = \mathcal{H}^\pm(z^\pm, \eta^\pm, \lambda) \quad (\text{C.8})$$

where $\mathcal{H}^\pm : C_b^0(J_\pm, \mathbb{R}^n) \times W_{t_0}^\pm \times \mathbb{R}^k \rightarrow C_b^0(J_\pm, \mathbb{R}^n)$. In order to apply the Implicit Function Theorem to the equation $z - \mathcal{H}^\pm(z^\pm, \eta^\pm, \lambda) = 0$ notice first that $\mathcal{H}^\pm(0, 0, 0) = 0$. On the other hand, $D_z \mathcal{H}^\pm(z^\pm, \eta^\pm, \lambda) : C_b^0(J_\pm, \mathbb{R}^n) \rightarrow C_b^0(J_\pm, \mathbb{R}^n)$ is the null function for $z^\pm = \eta^\pm = \lambda = 0$. Indeed, for any $h \in C_b^0(J_+, \mathbb{R}^n)$ it holds that

$$\begin{aligned} D_z \mathcal{H}^+(z, \eta, \lambda)h(t) &= \int_{t_0}^t \Phi(t, s) \mathcal{P}_+(s) D_z b(s, z(s), \lambda) h(s) ds \\ &\quad - \int_t^\infty \Phi(t, s) (I - \mathcal{P}_+(s)) D_z b(s, z(s), \lambda) h(s) ds. \end{aligned}$$

Since $D_z b(s, 0, 0) = 0$ then $D_z \mathcal{H}^+(0, 0, 0) = 0$. Similarly it follows that $D_z \mathcal{H}^-(0, 0, 0) = 0$. Therefore, there exists $\delta_\pm > 0$ such that for every $\eta^\pm \in W_{t_0}^\pm$ and $\lambda \in \mathbb{R}^k$ with $|\eta^\pm|, |\lambda| < \delta_\pm$ there is a unique $z^\pm(\eta^\pm, \lambda) \in C_b^0(J_\pm, \mathbb{R}^n)$ so that $z^\pm(\eta^\pm, \lambda) = \mathcal{H}^\pm(z^\pm(\eta^\pm, \lambda), \eta^\pm, \lambda)$ and $z^\pm(0, 0) = 0$.

Now, consider the condition $z^-(\lambda, \eta^-)(t_0) - z^+(\lambda, \eta^+)(t_0) \in E_{t_0}^*$. Since $Y_{t_0} = W_{t_0}^+ \oplus W_{t_0}^- \oplus E_{t_0}^*$, we can write

$$z^\pm(\eta^\pm, \lambda)(t_0) = \eta^\pm + w^\mp(\eta^\pm, \lambda) + \varrho^\pm(\eta^\pm, \lambda)$$

where $w^\mp(\eta^\pm, \lambda) \in W_{t_0}^\mp$ and $\varrho^\pm(\eta^\pm, \lambda) \in E_{t_0}^*$. From (C.6) and (C.7), and having into account that $X(t)X^{-1}(s)\mathcal{P}_\pm(s) = \mathcal{P}_\pm(t)X(t)X^{-1}(s)$, it follows that

$$\begin{aligned} z^-(\eta^-, \lambda)(t_0) &= \eta^- + \mathcal{P}_-(t_0) \int_{-\infty}^{t_0} X(t_0)X^{-1}(s)b(s, z^-(\eta^-, \lambda)(s), \lambda) ds, \\ z^+(\eta^+, \lambda)(t_0) &= \eta^+ - (I - \mathcal{P}_+(t_0)) \int_{t_0}^\infty X(t_0)X^{-1}(s)b(s, z^+(\eta^+, \lambda)(s), \lambda) ds. \end{aligned} \quad (\text{C.9})$$

Thus

$$\begin{aligned} w^+(\eta^-, \lambda) + \varrho^-(\eta^-, \lambda) &= \mathcal{P}_-(t_0) \int_{-\infty}^{t_0} X(t_0)X^{-1}(s)b(s, z^-(\eta^-, \lambda)(s), \lambda) ds, \\ w^-(\eta^+, \lambda) + \varrho^+(\eta^+, \lambda) &= -(I - \mathcal{P}_+(t_0)) \int_{t_0}^\infty X(t_0)X^{-1}(s)b(s, z^+(\eta^+, \lambda)(s), \lambda) ds. \end{aligned} \quad (\text{C.10})$$

Recall that $z^\pm(0, 0) = 0$ and hence $w^\pm(0, 0) = \varrho^\pm(0, 0) = 0$. On the other hand, applying that $D_z b(t, 0, 0) = 0$ in (C.10) we get that

$$\begin{aligned} D_{\eta^\mp} \varrho^\mp(0, 0) &= D_\lambda \varrho^\mp(0, 0) = 0, \\ D_{\eta^\mp} w^\pm(0, 0) &= D_\lambda w^\pm(0, 0) = 0. \end{aligned} \quad (\text{C.11})$$

The condition $z^-(\lambda, \eta^-)(t_0) - z^+(\lambda, \eta^+)(t_0) \in E_{t_0}^*$ is equivalent to the system of two equations $\eta^\pm - w^\pm(\eta^\mp, \lambda) = 0$ which we write in the form $F(\eta, \lambda) = 0$ where $\eta = (\eta^+, \eta^-) \in W_{t_0}^+ \times W_{t_0}^-$. We have that $F(0, 0) = 0$ and according to (C.11) it follows that $D_\eta F(0, 0) = I$. Hence, applying again the Implicit Function Theorem we get η as a function of λ . We conclude that there exists $\delta > 0$ ($\delta < \delta_\pm$) such that for any $\lambda \in \mathbb{R}^k$ with $|\lambda| < \delta$ there is a unique $\eta^\pm(\lambda) \in W_{t_0}^\pm$ so that $\eta^\pm(\lambda) - w^\pm(\eta^\mp(\lambda), \lambda) = 0$ and $\eta^\pm(0) = 0$.

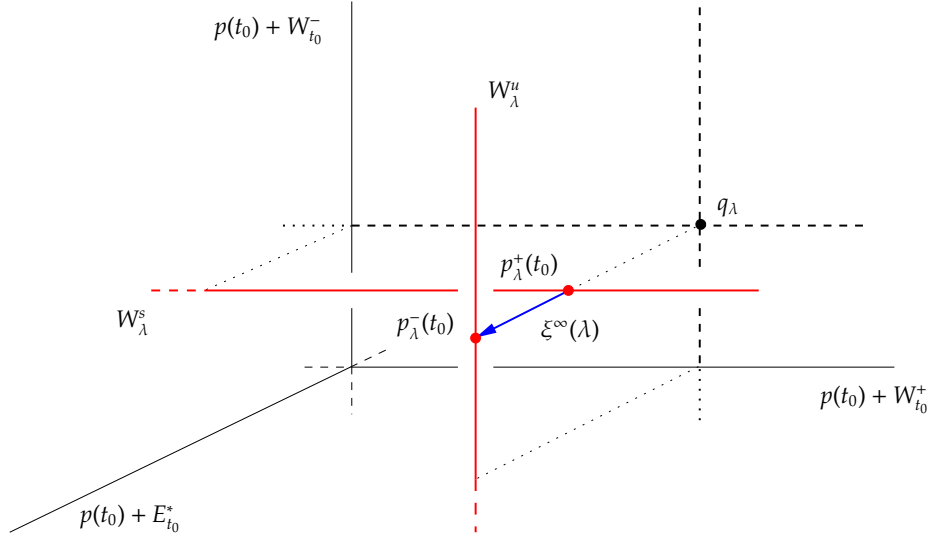


Figure 4. The figure shows the transversal section $\Sigma_{t_0} = p(t_0) + [W_{t_0}^+ \oplus W_{t_0}^- \oplus E_{t_0}^*]$ at the point $p(t_0)$ to a non degenerate (homo)heteroclinic orbit $\gamma = \{p(t) : t \in \mathbb{R}\}$. The curves W_λ^s and W_λ^u are, respectively, the intersections $W^s(p_+(\lambda)) \cap \Sigma_{t_0}$ and $W^u(p_-(\lambda)) \cap \Sigma_{t_0}$. A priori the curve W_λ^s does not meet W_λ^u . However, the projection along the direction $E_{t_0}^*$ of both curves on $p(t_0) + [W_{t_0}^- \oplus W_{t_0}^+]$ have a unique transversal intersection point q_λ . Now, $q_\lambda + E_{t_0}^*$ meets, respectively, W_λ^s and W_λ^u at $p_\lambda^+(t_0)$ and $p_\lambda^-(t_0)$. This two points define the vector $\xi^\infty(\lambda)$. The persistence of the connection holds if $\xi^\infty(\lambda) = 0$ which provides a set of $d = \dim E_{t_0}^*$ conditions.

Now consider the functions $z_\lambda^\pm(t) = z^\pm(\eta^\pm(\lambda), \lambda)(t)$. The uniqueness, boundedness and regularity respect to λ of $z_\lambda^\pm(t)$, and thus of $p_\lambda^\pm(t) = p(t) + z_\lambda^\pm(t)$, are following from the Implicit Function Theorem. Also we have that $z_0^\pm = 0$ and since for $|\lambda|$ small enough z_λ^\pm is close to $z_0^\pm = 0$ we get that $\sup_{t \in J_\pm} |p_\lambda^\pm(t) - p(t)|$ is arbitrarily small. This means that the orbits parameterize by $p_\lambda^\pm(t)$ are close to the orbit γ which is parameterized by $p(t)$. So, together the hyperbolicity of the equilibria $p_\pm(\lambda)$ we get that $\lim_{t \rightarrow \pm\infty} p_\lambda^\pm(t) = p_\pm(\lambda)$. Thus, for each λ with $|\lambda| < \delta$, the solutions $p_\lambda^\pm(t)$ parameterizes, respectively, orbits in the stable manifold $W^s(p_+(\lambda))$ and in the unstable manifold $W^u(p_-(\lambda))$ such that $p_\lambda^\pm(t_0) \in p(t_0) + Y_{t_0} = \Sigma_{t_0}$ and $\xi^\infty(\lambda) = p_\lambda^-(t_0) - p_\lambda^+(t_0) = z_\lambda^-(t_0) - z_\lambda^+(t_0) \in E_{t_0}^*$. This proves the first item of the lemma.

In order to prove the second item notice that if $\xi^\infty(\lambda) = 0$ then a (homo)heteroclinic orbit of (2.1) is given by

$$p_\lambda(t) = \begin{cases} p_\lambda^-(t) & \text{para } t \leq t_0, \\ p_\lambda^+(t) & \text{para } t \geq t_0. \end{cases}$$

where $p_\lambda^\pm(t)$ are the solutions in the first item. On the other hand, if $p_\lambda(t)$ is a solution parametrising a (homo)heteroclinic connection such that $p_\lambda(t_0) \in \Sigma_{t_0}$ with $|\lambda|$ and $|p_\lambda(t_0) - p(t_0)|$ small enough, its restriction to the intervals $J_- = (-\infty, t_0]$ and $J_+ = [t_0, \infty)$ define a pair of

solutions $p_\lambda^\pm(t)$ in the assumption of the first item. That is, $p_\lambda^\pm(t_0) \in \Sigma_{t_0}$ and $p_\lambda^-(t_0) - p_\lambda^+(t_0) = 0 \in E_{t_0}^*$. This makes obvious the reciprocal implication.

To conclude the proof of the second item notice that

$$\xi^\infty(\lambda) = p_\lambda^-(t_0) - p_\lambda^+(t_0) = z_\lambda^-(t_0) - z_\lambda^+(t_0) \in E_{t_0}^*.$$

Since $\{w_i : i = 1 \dots d\}$ is a base of $E_{t_0}^* = E_{t_0}^{s*} \cap E_{t_0}^{u*} = [E_{t_0}^s + E_{t_0}^u]^\perp$ then

$$\xi^\infty(\lambda) = \sum_{i=1}^d \langle w_i, \xi^\infty(\lambda) \rangle w_i.$$

Form (C.9) and having into account that $\langle w_i, \eta^\pm \rangle = 0$, it follows that

$$\begin{aligned} \xi_i^\infty(\lambda) &= \langle w_i, \xi^\infty(\lambda) \rangle = \langle w_i, \mathcal{P}_-(t_0) \int_{-\infty}^{t_0} X(t_0)X^{-1}(s)b(s, z_\lambda^-(s), \lambda) ds \rangle \\ &\quad + \langle w_i, (I - \mathcal{P}_+(t_0)) \int_{t_0}^{\infty} X(t_0)X^{-1}(s)b(s, z_\lambda^+(s), \lambda) ds \rangle \\ &= \int_{-\infty}^{t_0} \langle [\mathcal{P}_-(t_0)X(t_0)X^{-1}(s)]^* w_i, b(s, z_\lambda^-(s), \lambda) \rangle ds \\ &\quad + \int_{t_0}^{\infty} \langle [(I - \mathcal{P}_+(t_0))X(t_0)X^{-1}(s)]^* w_i, b(s, z_\lambda^+(s), \lambda) \rangle ds. \end{aligned} \tag{C.12}$$

Thus, since $\mathcal{P}_-(t_0)^* : \mathbb{R}^n \rightarrow E_{t_0}^{u*}$ and $I - \mathcal{P}_+(t_0)^* : \mathbb{R}^n \rightarrow E_{t_0}^{s*}$ we get that

$$[\mathcal{P}_-(t_0)X(t_0)X^{-1}(s)]^* w_i = X^{-1}(s)^* X(t_0)^* \mathcal{P}_-(t_0)^* w_i = w_i(s)$$

$$[(I - \mathcal{P}_+(t_0))X(t_0)X^{-1}(s)]^* w_i = X^{-1}(s)^* X(t_0)^* (I - \mathcal{P}_+(t_0)^*) w_i = w_i(s).$$

Substituting in (C.12) we obtain that

$$\xi_i^\infty(\lambda) \equiv \int_{-\infty}^{t_0} \langle w_i(s), b(s, z_\lambda^-(s), \lambda) \rangle ds + \int_{t_0}^{\infty} \langle w_i(s), b(s, z_\lambda^+(s), \lambda) \rangle ds = 0.$$

This concludes the second item and proves Lemma 2.16. \square

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